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Abstract. The electrical and thermal conduction properties of disordered solids and the possible degradation processes induced by the generation of cracks are central issues in the field of the heterogeneous materials. However, most of the existing theories are unable to consider an arbitrary density of cracks. We obtained an exact result for the fields induced within an elliptic anisotropic inhomogeneity embedded in a different anisotropic (two-dimensional) conductor. Then, we applied it to show that the degradation process strongly depends on the statistical orientational distribution of defects: in particular we theoretically prove that parallel cracks lead to the power law decay $\log \sigma \sim -\log N$ while random oriented cracks lead to the exponential law decay $\log \sigma \sim -N$ (where σ is the effective conductivity of a region with a large number N of defects), as recently predicted by numerical findings.

1 Introduction

The problem of finding the effective physical behaviour of heterogeneous, composite, disordered and defected materials plays a central role in material science and nanotechnology [1,2]. From the theoretical point of view this analysis can be conducted by means of the combination of two successive steps: the starting point is the determination of the physical behaviour of a single particle (or defect) embedded in a given matrix. Once this first problem is solved, homogenization procedures or effective medium theories can be applied in order to determine the overall effective properties [3,4]. They are typically implemented through some ad hoc averaging processes over the heterogeneous region. Both the problem of the single inhomogeneity and that of the heterogeneous structures (dispersions, polycrystals etc.) have been largely discussed both for electric and elastic properties [5-8].

The predictions of homogenization schemes are of primary importance in several fields, ranging from material science, to geophysics, to biology. For example, dispersions of cracks can strongly modify the response of fibre-reinforced materials [9,10]. Other interesting applications concern the behaviour of rocks, where cracking with different orientational distribution is originated by thermal gradients or tectonic stresses [11,12]. In addition, non isotropic fracture mechanisms are relevant in mineralized tissues like bone and dentin [13,14].

Despite the large number of works dealing with this subject and the wide interest of the material science community, the determination of the effective properties for dispersions with dense populations of inhomogeneities or defects is still severely limited both from the numerical and the theoretical point of view [1,2,6]. In fact, the interactions among the embedded particles are very complicated and their description, in many cases, is beyond the current methodologies [3,4,15]. Also for the specific case of multi-cracked materials several results can be found when the cracks density is not very high [16-18]. Moreover, this problem is even more difficult if the cracks can be oriented in different ways, generating complex anisotropic behaviours [19-21].

Recently, it has been observed that the macroscopic degradation of materials with a large number of cracks is governed by different laws depending on the orientational statistical distribution of the cracks within the host matrix [22–27]. More precisely, two different specific decays of the effective elastic constants have been observed for dispersions of uniformly random oriented cracks and parallel cracks. As a matter of fact, there are indications that a uniformly random oriented population of cracks leads to an exponential degradation of the elastic response in terms of the crack density [22,23] (followed by a percolative behaviour beyond a given percolation threshold) while a population of parallel cracks leads to a power-law decay of the effective response [24–27].

This scenario clearly indicates the need for a more detailed understanding of the degradation behaviour of multi-cracked materials. In fact, a model of general validity which explain the degradation behaviour versus the orientational distribution of cracks is not available in literature. This corresponds to the content the present paper where we analyse the conduction properties of a multicracked material with an arbitrary statistical orientation of defects (ranging from parallel to random distributions).

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Our approach is based on the two-step procedure above introduced: firstly, we obtain an exact result describing the electric field behaviour within an elliptic inhomogeneity embedded in an anisotropic two-dimensional conductor. Basically, the solution for the inhomogeneity has been obtained by the so-called equivalence principle which is formulated by an inclusion scheme described by a suitable eigenfield. The idea has been successfully adopted in many previous investigations as well, for the pure electric case [28], for the pure elastic case [29-32] and for the coupled piezoelectric case [33]. The important point of the present development lies in the combination of the anisotropy and the two-dimensionality of the system, which allow us to obtain a new closed form expression for the polarisation tensor of the particle. This is a strong advantage for the following applications since the complex integral expressions of earlier results are removed. The elliptic geometry allows us to use such a result to model a crack: if one of the principal axes of the ellipse becomes negligibly small, then the ellipse reduces to a thin crack. Of course, our results based on the elliptical shape, converge in the limiting case to a linear crack (represented by a segment on the plane). In some specific cases, different shapes have been considered in earlier literature (non-elliptic particles [34] and curvilinear cracks [35,36]).

As second step, we adopt an ad hoc iterative technique considering an increasing number of cracks, leading to a specific system of nonlinear differential equations for the effective conductivities of a multi-cracked material. By combining the anisotropic character of the system with the iterative scheme, we confirm the two different degradation behaviours of the effective conductivity for parallel and random geometries, already observed in literature [22–27]. Interestingly enough, we have obtained the complete description of the system for all the orientational distribution comprised between the two above limiting cases: in other words, we have obtained an unified formalism ranging from the power law decay to the exponential law decay. To conclude, this approach opens the possibility to take into consideration arbitrary populations of cracks and furnishes a physical understanding of the degradation process.

We underline that in this work we deal with the electric conduction properties while many previous investigations have treated the elastic response as well; nevertheless, a strong connection between the conductivity and the elastic moduli in composite structures is well known. This relationship has been formalized in earlier works in order to obtain some correlations among the effective elastic moduli and the effective conductivity of mixtures [37,38]. In particular, it has been exactly proved that the effective conductivity corresponds to the effective bulk modulus, if the Poisson ratios of the phases are set to zero [39].

The paper is structured in the following way: in Section 2, through the Green function for the two-dimensional electrostatics of anisotropic media, we introduce the concepts of eigenfield and inclusion. In Section 3 we explain the equivalence principle, which allows us to solve the anisotropic inhomogeneity problem. Finally, in Section 4, we show the application of the previous results to the electric characterization of anisotropic multi-cracked conductors. Firstly, we develop an homogenization theory which is valid only for a dilute dispersion of cracks; further, by means of the iterative technique, which implicitly take into account the interactions among cracks, we extend the applicability of the theory to higher values of the crack density. We finally present a simple method to obtain an estimate of the percolation threshold in the case of randomly oriented cracks.

2 Inclusions in anisotropic environment

The electrical behaviour of an inhomogeneity is a specific topic largely discussed in scientific literature: in fact, the basic solution for an isotropic ellipsoidal particle embedded in an isotropic environment, is a very well-known result [40,41]. The complete solution (written in integral form) for the linear and nonlinear anisotropic case has been recently investigated [28]. Moreover, the analysis of the electromagnetic behaviour of a bianisotropic inhomogeneity has been performed by means of the so-called depolarization tensor [42–45]. From the mechanical point of view, the problem of an elastic inhomogeneity has been solved by Eshelby by means of a very elegant mathematical procedure [29-32]. Such a formalism has been further generalised to the case of a nonlinear inhomogeneity [46,47] and many attempts have been proposed to describe scale effects in nano-particles [48–50].

The first aim of the present work is to discuss an exact result concerning the response of an elliptic particle embedded in a two-dimensional conducting material. Although the general problem has been largely discussed in literature as reported above, we have introduced here a methodology to cope with the completely anisotropic problem (with different conductivity tensors in the regions inside and outside the particle) in order to obtain an explicit result in algebraic form. We remark that all the previous results concerning anisotropic particles were written in integral form [28,42,43]: the consideration of the two-dimensional system allows us to remove complicated integrals from the solutions and our achievement is very convenient for analysing paradigmatic problems related to heterogeneous and composite structures. At the same time, however, we can take into account any possible anisotropy and aspect ratio for the particle. The explicit expression for the electric field inside the elliptic particle is very useful for the subsequent applications dealing with the multi-defected two-dimensional conductor.

A linear anisotropic electric conductor is described by the standard constitutive equation $J = \hat{\sigma} E$ where J is the current density vector, E is the electric field and $\hat{\sigma}$ is the conductivity tensor. We define a certain region of the plane as an *inclusion* when the constitutive equation, in that zone, assumes the form $J = \hat{\sigma} (E - E^*)$ where $E^*(x)$ is an assigned vector function of the position x, which is named *eigenfield* (see Fig. 1). We remark that the concept of inclusion is determined by the presence of a given eigenfield, which modifies the constitutive equation as above discussed, but it is not connected with the conductivity tensor $\hat{\sigma}$, which remains homogeneous in the entire plane. In this work, in order to indicate spatial variations of the conductivity tensor, we adopt the term *inhomogeneity*. The problem of an inclusion discussed in the present section will be used to solve the problem of the inhomogeneity in the next section.

The eigenfield, defined in some region of the plane, acts as a sort of source and its effects can be studied as follows. In a steady state regime we may use the continuity equation $\nabla \cdot \boldsymbol{J} = 0$ obtaining $\nabla \cdot [\hat{\sigma} (\boldsymbol{E} - \boldsymbol{E}^*)] = 0$ or, equivalently, $\nabla \cdot [\hat{\sigma} \boldsymbol{E}] = \nabla \cdot [\hat{\sigma} \boldsymbol{E}^*]$. Now, we can introduce the potential in the standard way, writing the relation $\nabla \cdot [\hat{\sigma} \nabla V(\boldsymbol{r})] = -\nabla \cdot [\hat{\sigma} \boldsymbol{E}^*]$ or, similarly, the generalized Poisson equation

$$\sigma_{ij}\frac{\partial^2 V\left(\boldsymbol{x}\right)}{\partial x_i \partial x_j} = -s\left(\boldsymbol{x}\right) \tag{1}$$

where $s = \nabla \cdot [\hat{\sigma} E^*] = \sigma_{ij} \frac{\partial E_j^*(x)}{\partial x_i}$ is the source equivalent to the eigenfield and we have adopted the Einstein summation rule on the repeated indices. So, we want to analyse the effects of the presence of a given inclusion (described by its eigenfield). To begin, we suppose that the eigenfield is defined in the whole two-dimensional space and, therefore, we may solve the generalized Poisson equation (see Eq. (1)) by means of the Green function defined by

$$\sigma_{ij}\frac{\partial^2 G\left(\boldsymbol{x}\right)}{\partial x_i \partial x_j} = -\delta\left(\boldsymbol{x}\right). \tag{2}$$

We can use the standard two-dimensional Fourier transform converting the function $G(\boldsymbol{x})$ to $\mathcal{G}(\boldsymbol{\omega})$. We simply obtain

$$\mathcal{G}\left(\boldsymbol{\omega}\right) = \frac{1}{\omega_i \sigma_{ij} \omega_j} \tag{3}$$

and therefore

$$G(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_{\Re^2} \frac{1}{\omega_i \sigma_{ij} \omega_j} e^{i\boldsymbol{\omega} \cdot \boldsymbol{x}} d\boldsymbol{\omega}.$$
 (4)

The solution of equation (1) can be written as the convolution

$$V(\boldsymbol{x}) = \int_{\Re^2} s(\boldsymbol{x}') G(\boldsymbol{x} - \boldsymbol{x}') d\boldsymbol{x}'$$
$$= \sigma_{ij} \int_{\Re^2} \frac{\partial E_j^*(\boldsymbol{x}')}{\partial x_i'} G(\boldsymbol{x} - \boldsymbol{x}') d\boldsymbol{x}'.$$
(5)

Now, we can use an integration by part, holding on for multiple integrals, which can be written as follows

$$\int_{\Re^2} \vartheta\left(\boldsymbol{x}\right) \frac{\partial \lambda\left(\boldsymbol{x}\right)}{\partial x_l} d\boldsymbol{x} = -\int_{\Re^2} \lambda\left(\boldsymbol{x}\right) \frac{\partial \vartheta\left(\boldsymbol{x}\right)}{\partial x_l} d\boldsymbol{x}$$
(6)

where $\vartheta(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ are two given functions with sufficiently regular behaviour at infinity (this property is an immediate consequence of the Gauss-Ostrogradsky theorem). The application of equation (6) to equation (5) simply leads to

$$V(\boldsymbol{x}) = -\sigma_{ij} \int_{\Re^2} E_j^*(\boldsymbol{x}') \frac{\partial G(\boldsymbol{x} - \boldsymbol{x}')}{\partial x_i'} d\boldsymbol{x}'$$
$$= \sigma_{ij} \int_{\Re^2} E_j^*(\boldsymbol{x}') \frac{\partial G(\boldsymbol{x} - \boldsymbol{x}')}{\partial x_i} d\boldsymbol{x}'.$$
(7)

If the eigenfield $E_k^*(\boldsymbol{x})$ is constant in a limited region Ω of the plane we can say that we are dealing with a *uniform* or *homogeneous inclusion* Ω and the electric potential over the entire plane become

$$V(\boldsymbol{x}) = \sigma_{ij} E_j^* \int_{\Omega} \frac{\partial G(\boldsymbol{x} - \boldsymbol{x}')}{\partial x_i} d\boldsymbol{x}'$$

$$= \frac{1}{4\pi^2} \sigma_{ij} E_j^* \frac{\partial}{\partial x_i} \int_{\Omega} \int_{\Re^2} \frac{e^{i\boldsymbol{\omega} \cdot (\boldsymbol{x} - \boldsymbol{x}')}}{\omega_k \sigma_{kh} \omega_h} d\boldsymbol{\omega} d\boldsymbol{x}'$$

$$= \frac{1}{4\pi^2} \sigma_{ij} E_j^* \frac{\partial}{\partial x_i} \int_{\Re^2} \frac{e^{i\boldsymbol{\omega} \cdot \boldsymbol{x}}}{\omega_k \sigma_{kh} \omega_h} Q(\boldsymbol{\omega}) d\boldsymbol{\omega} \qquad (8)$$

where

$$Q(\boldsymbol{\omega}) = \int_{\Omega} e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}'} d\boldsymbol{x}'.$$
 (9)

We suppose now that the region Ω is elliptic and described by $\Omega = \{ \boldsymbol{x} : \boldsymbol{x} \cdot \hat{a}^{-2} \boldsymbol{x} \leq 1 \}$, where \hat{a} is a symmetric and positive definite matrix $(a_{11} > 0, a_{22} > 0 \text{ and } a_{11}a_{22} - a_{12}^2 > 0)$. For sake of generality we consider an arbitrarily oriented ellipse on the plane and, therefore, we may have $a_{12} \neq 0$ (see Fig. 1). With the change of variable $\boldsymbol{x}' = \hat{a}\boldsymbol{y}$ the integral for $Q(\boldsymbol{\omega})$ assumes the simpler form

$$Q(\boldsymbol{\omega}) = \det \hat{a} \int_{\|\boldsymbol{y}\| \le 1} e^{-i\boldsymbol{y}\cdot\hat{a}\boldsymbol{\omega}} d\boldsymbol{y}.$$
 (10)

Now we can adopt the standard polar coordinates $y = (R\cos\theta, R\sin\theta)$ on the plane and the exponent term in equation (10) can be written as

$$\boldsymbol{y} \cdot \hat{a}\boldsymbol{\omega} = R\sqrt{\boldsymbol{\omega} \cdot \hat{a}^{2}\boldsymbol{\omega}} \left(\frac{(\hat{a}\boldsymbol{\omega})_{1}\cos\theta}{\sqrt{\boldsymbol{\omega} \cdot \hat{a}^{2}\boldsymbol{\omega}}} + \frac{(\hat{a}\boldsymbol{\omega})_{2}\sin\theta}{\sqrt{\boldsymbol{\omega} \cdot \hat{a}^{2}\boldsymbol{\omega}}} \right)$$
$$= R\sqrt{\boldsymbol{\omega} \cdot \hat{a}^{2}\boldsymbol{\omega}}\cos\left(\theta - \theta_{0}\right) \tag{11}$$

where the angle θ_0 is defined through $\cos \theta_0 = \frac{(\hat{a}\omega)_1}{\sqrt{\omega \cdot \hat{a}^2 \omega}}$ and $\sin \theta_0 = \frac{(\hat{a}\omega)_2}{\sqrt{\omega \cdot \hat{a}^2 \omega}}$. By letting $\psi = \theta - \theta_0$ we simply obtain

$$Q(\boldsymbol{\omega}) = \det \hat{a} \int_0^{2\pi} \int_0^1 e^{-iR\sqrt{\boldsymbol{\omega}\cdot\hat{a}^2\boldsymbol{\omega}}\cos\psi} RdRd\psi.$$
(12)



Fig. 1. (Color online) Schematic representation of the matrix/inclusion system defined by the eigenfield E^* uniformly distributed inside the ellipse Ω .

We can now use the following integral representation of the $J_0(z)$ Bessel function [51]

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\cos\psi} d\psi.$$
 (13)

Thus, we have

$$Q(\boldsymbol{\omega}) = 2\pi \det \hat{a} \int_0^1 J_0\left(R\sqrt{\boldsymbol{\omega} \cdot \hat{a}^2 \boldsymbol{\omega}}\right) R dR \qquad (14)$$

and finally

$$Q(\boldsymbol{\omega}) = 2\pi \det \hat{a} \frac{J_1\left(\sqrt{\boldsymbol{\omega} \cdot \hat{a}^2 \boldsymbol{\omega}}\right)}{\sqrt{\boldsymbol{\omega} \cdot \hat{a}^2 \boldsymbol{\omega}}}$$
(15)

where we have used the standard properties of the Bessel function $J_1(z)$ [51]. So, coming back to equation (8) we calculate the electric field induced by the uniform inclusion in the entire plane as $E_k(\mathbf{x}) = -\partial V(\mathbf{x}) / \partial x_k$, by obtaining

$$E_k = \frac{\det \hat{a}}{2\pi} \sigma_{ij} E_j^* \int_{\Re^2} \frac{\omega_i \omega_k e^{i\boldsymbol{\omega} \cdot \boldsymbol{x}}}{\boldsymbol{\omega} \cdot \hat{\sigma} \boldsymbol{\omega}} \frac{J_1\left(\sqrt{\boldsymbol{\omega} \cdot \hat{a}^2 \boldsymbol{\omega}}\right)}{\sqrt{\boldsymbol{\omega} \cdot \hat{a}^2 \boldsymbol{\omega}}} d\boldsymbol{\omega}.$$
(16)

Such an integral can be handled by the introduction of the polar coordinates $\boldsymbol{\omega} = (\omega \cos \alpha, \omega \sin \alpha)$ and the definition of the unit vector $\boldsymbol{n} = (\cos \alpha, \sin \alpha)$. With these settings we have $\boldsymbol{\omega} = \omega \boldsymbol{n}$ and $d\boldsymbol{\omega} = \omega d\omega d\alpha$ and equation (16) becomes

$$E_k(\boldsymbol{x}) = \frac{\det \hat{a}}{2\pi} \sigma_{ij} E_j^* \int_0^{2\pi} \frac{n_i n_k}{\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}} \frac{\mathcal{K}(\alpha)}{\sqrt{\boldsymbol{n} \cdot \hat{a}^2 \boldsymbol{n}}} d\alpha \quad (17)$$

where the integral ${\cal K}$ contains all the terms depending on ω and it is is defined below

$$\mathcal{K}(\alpha) = \int_0^\infty e^{i\omega\boldsymbol{n}\cdot\boldsymbol{x}} J_1\left(\omega\sqrt{\boldsymbol{n}\cdot\hat{a}^2\boldsymbol{n}}\right) d\omega.$$
(18)

By using formulas 6.671 (1) and (2) of the Gradshteyn-Ryzhik table of integrals [52] it is simple to verify that, if $\boldsymbol{x} \cdot \hat{a}^{-2}\boldsymbol{x} \leq 1$ (inside the inclusion), then we simply have $\Re e \{\mathcal{K}\} = 1/\sqrt{\boldsymbol{n} \cdot \hat{a}^2 \boldsymbol{n}}$, independently of \boldsymbol{x} . It means that inside the inclusion the induced electric field is uniform and its components are given by

$$E_k = \mathcal{S}_{kj} E_j^* \tag{19}$$

where \hat{S} is the electric counterpart of the Eshelby tensor [29,30] for the two-dimensional anisotropic conduction. It is given by

$$\hat{S} = \frac{\det \hat{a}}{2\pi} \int_0^{2\pi} \frac{(\boldsymbol{n} \otimes \boldsymbol{n}) \,\hat{\sigma}}{(\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}) \,(\boldsymbol{n} \cdot \hat{a}^2 \boldsymbol{n})} d\alpha \tag{20}$$

where we have defined the tensor product of vector as $(\boldsymbol{n} \otimes \boldsymbol{n})_{ki} = n_k n_i$. It is important to observe that the quantity

$$\frac{\hat{S}\hat{\sigma}^{-1}}{\det\hat{a}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\boldsymbol{n}\otimes\boldsymbol{n})}{(\boldsymbol{n}\cdot\hat{\sigma}\boldsymbol{n})(\boldsymbol{n}\cdot\hat{a}^2\boldsymbol{n})} d\alpha \qquad (21)$$

is a symmetric functional of the two tensors $\hat{\sigma}$ and \hat{a}^2 by construction. This last integral can be evaluated through the following key result which is proved in Appendix A (where we will consider $\hat{\eta} = \hat{a}^2$)

$$\frac{1}{2\pi} \int_{0}^{2\pi} \frac{\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}) (\boldsymbol{n} \cdot \hat{\eta} \boldsymbol{n})} d\alpha = \frac{\hat{\sigma}^{-1} \sqrt{\det \hat{\sigma}} + \hat{\eta}^{-1} \sqrt{\det \hat{\eta}}}{\operatorname{tr} \hat{\sigma} \operatorname{tr} \hat{\eta} - \operatorname{tr} (\hat{\sigma} \hat{\eta}) + 2\sqrt{\det \hat{\sigma}} \sqrt{\det \hat{\eta}}}.$$
(22)

The explicit form of the tensor \hat{S} can be now written in a very convenient form through the following considerations. The characteristic polynomial of the matrix $\hat{\eta}$ can be written as

$$\det\left(\hat{\eta} - \lambda\hat{I}\right) = \det\hat{\eta} - \lambda \mathrm{tr}\hat{\eta} + \lambda^2 \tag{23}$$

and, therefore, by applying the Cayley-Hamilton theorem (which states that every square matrix satisfies its own characteristic polynomial) we can write the relation $\det \hat{\eta} - \hat{\eta} \operatorname{tr} \hat{\eta} + \hat{\eta}^2 = 0$ or, equivalently, $\hat{\eta}^{-1} \det \hat{\eta} = \hat{I} \operatorname{tr} \hat{\eta} - \hat{\eta}$. Consequently, we obtain the result

$$\operatorname{tr}\hat{\sigma}\operatorname{tr}\hat{\eta} - \operatorname{tr}\left(\hat{\sigma}\hat{\eta}\right) = \operatorname{tr}\left[\hat{\sigma}\operatorname{tr}\hat{\eta} - \hat{\sigma}\hat{\eta}\right]$$
$$= \operatorname{tr}\left[\hat{\sigma}\left(\hat{I}\operatorname{tr}\hat{\eta} - \hat{\eta}\right)\right] = \operatorname{det}\hat{\eta}\operatorname{tr}\left(\hat{\eta}^{-1}\hat{\sigma}\right)$$
(24)

which can be used in the denominator of equation (22). Finally, by combining equations (21), (22) and (24) we obtain the result

$$\hat{S} = \frac{\hat{a}^{-2}\hat{\sigma} + \hat{I}\sqrt{\det\left(\hat{a}^{-2}\hat{\sigma}\right)}}{\operatorname{tr}\left(\hat{a}^{-2}\hat{\sigma}\right) + 2\sqrt{\det\left(\hat{a}^{-2}\hat{\sigma}\right)}}.$$
(25)

This is the most important achievement of the present Section: when an uniform inclusion of elliptic shape



Fig. 2. (Color online) Schematic representation of the matrix/inhomogeneity system defined by the different conductivity tensors $\hat{\sigma}$ and $\hat{\sigma}_i$, outside and inside the ellipse Ω , respectively.

 $\Omega = \{ \boldsymbol{x} : \boldsymbol{x} \cdot \hat{a}^{-2}\boldsymbol{x} \leq 1 \}$ and eigenfield \boldsymbol{E}^* is embedded in an anisotropic environment of conductivity tensor $\hat{\sigma}$, inside the particle a uniform electric field $\boldsymbol{E} = \hat{S}\boldsymbol{E}^*$ is induced, where the Eshelby tensor \hat{S} is given in equation (25). Interestingly enough, we observe that the behaviour of the inclusion is completely controlled by the product of tensors $\hat{a}^{-2}\hat{\sigma}$. We remark that equation (25) is very useful for practical calculations since it is in a pure algebraic form. We also underline that all the expressions, previously published for the anisotropic Eshelby tensor, are in integral form [42,43]. When $\hat{\sigma} = \sigma \hat{I}$ (isotropic environment) and $a_{12} = 0$ (ellipse with axes aligned to the reference frame) we obtain

$$\hat{\mathcal{S}} = \begin{pmatrix} \frac{a_{22}}{a_{11} + a_{22}} & 0\\ 0 & \frac{a_{11}}{a_{11} + a_{22}} \end{pmatrix}$$
(26)

which is a classical result largely discussed in literature [40,41,58].

3 Equivalence principle

In this Section we show that the previous theory for an inclusion is useful to solve the problem of a given anisotropic inhomogeneity placed in a different anisotropic matrix. In fact, we will show an equivalence principle that reduces the analysis of the inhomogeneity behaviour to that of an inclusion. We consider an elliptic inhomogeneity with conductivity tensor $\hat{\sigma}_i$ embedded in an anisotropic environment with conductivity $\hat{\sigma}$ (see Fig. 2). We suppose that the system is subjected to a remote uniform electric field E^{∞} . We prove that the presence of an inhomogeneity can be described by the superimposition of two different situations A and B. The situation A is very simple because it considers the effects of the remote field E^{∞} in an homogeneous matrix without the inhomogeneity. In such a case, we simply observe that the density current vector

 $J^{\infty} = \hat{\sigma} E^{\infty}$ remains uniform in the entire plane. The situation B corresponds to an inclusion scheme where the eigenfield E^* is unknown and it can be determined by imposing the equivalence between the original problem and the superimposition A+B. We define J_{tot} and E_{tot} as the electric quantities in the inhomogeneity problem; the fields J^{∞} and E^{∞} completely describe the situation A; finally, the problem B is described by the electric variables J and E. Therefore, we have the superimpositions $J_{tot} = J^{\infty} + J$ and $E_{tot} = E^{\infty} + E$. Hence, inside the ellipsoid we obtain

$$\hat{\sigma}_i \boldsymbol{E}_{tot} = \hat{\sigma} \boldsymbol{E}^{\infty} + \hat{\sigma} \left(\boldsymbol{E} - \boldsymbol{E}^* \right)$$

$$\boldsymbol{E}_{tot} = \boldsymbol{E}^{\infty} + \boldsymbol{E}.$$
(27)

These relationships allow us to calculate the exact value of the eigenfield E^* that assure the equivalence between the initial problem and the model A+B. Since $E = \hat{S}E^*$ for $x \in \Omega$ (see Eq. (19)), we may write

$$\hat{\sigma}_{i}\boldsymbol{E}_{tot} = \hat{\sigma}\boldsymbol{E}^{\infty} + \hat{\sigma}\left(\hat{\mathcal{S}} - \hat{I}\right)\boldsymbol{E}^{*}$$

$$\boldsymbol{E}_{tot} = \boldsymbol{E}^{\infty} + \hat{\mathcal{S}}\boldsymbol{E}^{*}.$$
(28)

By substituting the second relation in the first one we have

$$\hat{\sigma}_i \left(\boldsymbol{E}^{\infty} + \hat{\boldsymbol{S}} \boldsymbol{E}^* \right) = \hat{\sigma} \boldsymbol{E}^{\infty} + \hat{\sigma} \left(\hat{\boldsymbol{S}} - \hat{I} \right) \boldsymbol{E}^*.$$
(29)

This is an equation in the eigenfield E^* that can be easily solved by obtaining

$$\boldsymbol{E}^* = \left[\left(\hat{I} - \hat{\sigma}^{-1} \hat{\sigma}_i \right)^{-1} - \hat{\mathcal{S}} \right]^{-1} \boldsymbol{E}^{\infty}.$$
(30)

This is the value of the eigenfield that ensures the validity of the equivalence principle. Moreover, we calculate the total electric field \boldsymbol{E}_{tot} induced inside the inhomogeneity. From the second relation given in equation (28) we derive $\boldsymbol{E}^* = \hat{\boldsymbol{S}}^{-1} (\boldsymbol{E}_{tot} - \boldsymbol{E}^{\infty})$ and therefore, from the first one, we have

$$\hat{\sigma}_i \boldsymbol{E}_{tot} = \hat{\sigma} \boldsymbol{E}^{\infty} + \hat{\sigma} \left(\hat{\mathcal{S}} - \hat{I} \right) \hat{\mathcal{S}}^{-1} \left(\boldsymbol{E}_{tot} - \boldsymbol{E}^{\infty} \right). \quad (31)$$

This equation in the unknown E_{tot} can be solved with straightforward algebraic calculations, arriving at the solution

$$\boldsymbol{E}_{tot} = \left[\hat{I} - \hat{\mathcal{S}}\left(\hat{I} - \hat{\sigma}^{-1}\hat{\sigma}_i\right)\right]^{-1} \boldsymbol{E}^{\infty}$$
(32)

when $x \in \Omega$. This is the internal electric field induced in the elliptic defect: this is a uniform vector field since all the quantities involved in equation (32) are constants. To conclude, the combination of equations (25) and (32) solves the problem of determining the electric field inside the inhomogeneity in term of the remotely applied field.

We remark that equation (32) is a standard relation well known in literature and it appears whenever one applies the equivalence principle as stated by Eshelby in his seminal work [29]. This approach is widely utilized ranging from electromagnetism (where it is combined with the method of moments [53,54]) and mechanics (where it is the base in most homogenization schemes [5,6]). Here we have reported a brief proof of equation (32) simply for giving a direct connection between the new result stated in equation (25) and the applications discussed in the following section.

4 Dispersions of cracks

Typically, when the response of a given inhomogeneity or defect is known, an averaging process must be chosen for describing the behaviour of a dispersion of particles in a given matrix: the coupling of the two steps generates a specific effective medium theory. Probably the first attempts to model a mixture is given by the Maxwell theory dealing with a very diluted suspension of conducting spheres [6,7,55]. A generalization to higher concentrations is provided by the differential method [56], which can be also applied to mixtures of ellipsoidal particles [57–59]. An approach based on the equivalent inclusion method has been developed in the framework of the steady-state thermal conduction [60-62]. Moreover, some numerical methods have been developed for considering objects having arbitrary shape [63]. Recent developments include nonrandom orientational distributions of ellipsoids [64], variational principles for multiple inclusions [65], nonlinear anisotropic mixtures [66] and methods for shape-distributed particles [67]. From the point of view of the mechanical homogenization theories, some similar investigations can be found in literature [68,69].

4.1 Dilute model

In our model the single defect is given by a crack on the plane. From the electrical point of view a crack in a two dimensional structure is a segment where the electric current cannot flow. In the present work, in order to model the flat shape of a crack, we adopt an elliptic void (zero conductivity, $\hat{\sigma}_i = 0$ and $\boldsymbol{J}_{tot} = 0$) with an axis with infinitesimal length. Treating the crack as a vacuous ellipse of aspect ratio approaching zero is very convenient. In fact, we can derive the needed formulas for a cracked solid through the limits in the general formulas, concerning elliptic inhomogeneities (see previous section). We theoretically analyse the effects of a given distribution of cracks on the conductivity of a solid. In earlier literature many works have been devoted to the study of this topic. In such works the orientational distribution of cracks is typically given by one of the two most adopted distributions: cracks aligned with a given direction and cracks uniformly oriented in the space (see introduction). Here we study a cracked solid with an arbitrary angular distribution of cracks and arbitrary crack density. The limiting cases of the present theory are represented by all the defects aligned with a given direction (order) and all the defects randomly oriented (disorder). We take into account all the intermediate configurations between order and disorder with the aim of characterising a material with cracks partially aligned.



Fig. 3. (Color online) Schematic representation of several dispersions of cracks in the anisotropic conductor with conductivity tensor $\hat{\sigma}$. Each sample is characterized by a different order parameter *P* corresponding to a particular orientational distribution of cracks.

In Figure 3 one can find some orientational distributions between the upon-described limiting cases. As discussed below, the angular distribution of cracks is statistically well described by an order parameter P.

We start by considering an elliptic region defined by $\Omega_0 = \{ \boldsymbol{x} : \boldsymbol{x} \cdot \hat{a}_0^{-2} \boldsymbol{x} \leq 1 \}$ with the axes a_{11} and a_{22} aligned to the reference frame, so that

$$\hat{a}_0 = \begin{pmatrix} a_{11} & 0\\ 0 & a_{22} \end{pmatrix}. \tag{33}$$

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Then, we apply a standard rotation matrix \mathcal{R} in order to obtain an arbitrarily oriented ellipse Ω described by

$$\hat{a}^{-2} = \hat{\mathcal{R}}^T \hat{a}_0^{-2} \hat{\mathcal{R}} \tag{34}$$

where $\hat{\mathcal{R}}$ depends on the rotation angle θ as follows

$$\hat{\mathcal{R}} = \begin{pmatrix} \cos\theta - \sin\theta\\ \sin\theta & \cos\theta \end{pmatrix}.$$
 (35)

The average value (over the angle θ) of the electric field induced inside a random oriented void defect can be obtained through the averaging of equation (32) with $\hat{\sigma}_i = 0$

$$\langle \boldsymbol{E}_{tot} \rangle = \left\langle \left[\hat{I} - \hat{S} \right]^{-1} \right\rangle_{\theta} \boldsymbol{E}^{\infty} \doteq \hat{\mathcal{W}} \boldsymbol{E}^{\infty}.$$
 (36)

In equation (36) we have also defined the concentration factor $\hat{\mathcal{W}}$, which will be very useful for the following developments of the homogenization theory. From equation (25) we simply obtain the tensor $\hat{I} - \hat{S}$ as follows

$$\hat{I} - \hat{S} = \frac{\hat{I} \left[\operatorname{tr} \left(\hat{a}^{-2} \hat{\sigma} \right) + \sqrt{\operatorname{det} \left(\hat{a}^{-2} \hat{\sigma} \right)} \right] - \hat{a}^{-2} \hat{\sigma}}{\operatorname{tr} \left(\hat{a}^{-2} \hat{\sigma} \right) + 2\sqrt{\operatorname{det} \left(\hat{a}^{-2} \hat{\sigma} \right)}} \quad (37)$$

and its inverse as

$$\left[\hat{I} - \hat{\mathcal{S}}\right]^{-1} = \hat{I} + \frac{\hat{a}^{-2}\hat{\sigma}}{\sqrt{\det\left(\hat{a}^{-2}\hat{\sigma}\right)}}.$$
(38)

Therefore, the tensor $\hat{\mathcal{W}}$ defined in equation (36) can be written in the form

$$\hat{\mathcal{W}} = \hat{I} + \frac{a_{11}a_{22}}{\sqrt{\det \hat{\sigma}}} \left\langle \left. \hat{\mathcal{R}}^T \hat{a}_0^{-2} \hat{\mathcal{R}} \right\rangle_{\theta} \hat{\sigma}.$$
(39)

Now, we have to calculate the average $\left\langle \hat{\mathcal{R}}^T \hat{a}_0^{-2} \hat{\mathcal{R}} \right\rangle_{\theta}$. The angle θ assumes the role of a random variable symmetrically distributed over the range $(-\pi/2, -\pi/2)$. The symmetry of the probability density assures that the average value of the product $\sin \theta \cos \theta$ (appearing in the product $\hat{\mathcal{R}}^T \hat{a}_0^{-2} \hat{\mathcal{R}}$) is exactly zero and the result depends only on the average value of $\cos^2 \theta$. Therefore, it is useful to define the following order parameter, which completely describes the state of order/disorder of the distribution of cracks

$$P = \left\langle 1 - \cos^2 \theta \right\rangle_{\theta} = \left\langle \sin^2 \theta \right\rangle_{\theta}.$$
 (40)

It is easy to observe that P assumes special values for particular angular distributions of cracks: if P = 0 all the cracks are parallel to the x_1 -axes (horizontal order), if P = 1 all the cracks are parallel to the x_2 -axes (vertical order) and if P = 1/2 the angle of rotation is uniformly distributed leading to a state of complete disorder. The other values cover all the orientational distribution between the random and the parallel ones (see Fig. 3 for some examples). Eventually, the definition of P allows us to write

$$\left\langle \hat{\mathcal{R}}^T \hat{a}_0^{-2} \hat{\mathcal{R}} \right\rangle_{\theta} = \begin{pmatrix} \frac{1-P}{a_{11}^2} + \frac{P}{a_{22}^2} & 0\\ 0 & \frac{P}{a_{11}^2} + \frac{1-P}{a_{22}^2} \end{pmatrix}.$$
 (41)

Therefore, equations (39) and (41) completely define the mathematical form of the tensor $\hat{\mathcal{W}}$.

We are now ready to work out the procedure aimed at defining the effective behaviour of a multi-cracked material. We consider a region S containing N voids (i.e. cracks). The area of a single ellipse is $\pi a_{11}a_{22} = \pi a^2 e$, where $a = a_{11}$ is the half-axis along the x_1 direction and $e = a_{22}/a_{11}$ is the aspect ratio. Therefore, the volume fraction of the voids is given by $c = \pi \frac{N}{S}a^2 e$. We may calculate the average value of the electric field over the multi-cracked material by means of the relation

$$\langle \boldsymbol{E} \rangle = c \langle \boldsymbol{E}_{tot} \rangle + (1-c) \boldsymbol{E}^{\infty} = \left[(1-c)\hat{I} + c\hat{\mathcal{W}} \right] \boldsymbol{E}^{\infty}$$
(42)

where, assuming a small crack density, we have considered the average electric field outside the inclusions similar to the remote field \mathbf{E}^{∞} . In this approximation (corresponding to a regime of non-interacting voids) each crack is subjected to the same external load \mathbf{E}^{∞} and it is not affected by other neighbouring cracks. We will generalize the procedure to larger values of the crack density in further steps. We, therefore, define $\hat{\sigma}_{eff}$ as the effective conductivity tensor of the whole anisotropic mixture by means of the relation $\langle \mathbf{J} \rangle = \hat{\sigma}_{eff} \langle \mathbf{E} \rangle$. In order to evaluate $\hat{\sigma}_{eff}$ we must calculate the average value $\langle \mathbf{J} \rangle$ of the stress tensor inside the multi-cracked material. We distinguish the total area S of the system, the area S_e of the embedded elliptic voids (i.e. the cracks) and the area S_o of the remaining space among the voids. We get

$$\langle \boldsymbol{J} \rangle = \frac{1}{S} \int_{S} \boldsymbol{J} d\boldsymbol{x} = \frac{1}{S} \hat{\sigma} \int_{S_{o}} \boldsymbol{E} d\boldsymbol{x}$$

$$= \frac{1}{S} \hat{\sigma} \int_{S_{o}} \boldsymbol{E} d\boldsymbol{x} + \frac{1}{S} \hat{\sigma} \int_{S_{e}} \boldsymbol{E} d\boldsymbol{x} - \frac{1}{S} \hat{\sigma} \int_{S_{e}} \boldsymbol{E} d\boldsymbol{x}$$

$$= \frac{1}{S} \hat{\sigma} \int_{S} \boldsymbol{E} d\boldsymbol{x} - \frac{S_{e}}{S} \hat{\sigma} \frac{1}{S_{e}} \int_{S_{e}} \boldsymbol{E} d\boldsymbol{x}$$

$$= \hat{\sigma} \langle \boldsymbol{E} \rangle - c \hat{\sigma} \langle \boldsymbol{E}_{tot} \rangle = \hat{\sigma} \langle \boldsymbol{E} \rangle - c \hat{\sigma} \hat{\mathcal{W}} \boldsymbol{E}^{\infty}.$$

$$(43)$$

Drawing a comparison between equations (42) and (43) we may find the expression for the effective conductivity tensor

$$\hat{\sigma}_{eff} = \hat{\sigma} \left\{ \hat{I} - c\hat{\mathcal{W}} \left[(1-c)\hat{I} + c\hat{\mathcal{W}} \right]^{-1} \right\}.$$
(44)

We remark that this result is equivalent to the standard Mori-Tanaka formalism [6]. In order to recover the shape of real cracks, we need to elaborate the above formulas in the limit of vanishing aspect ratio. Since the limit for e approaching zero is equivalent to the limit for c approaching zero, we get

$$\lim_{e \to 0} c\hat{\mathcal{W}} = \lim_{c \to 0} c\hat{\mathcal{W}} = \hat{\mathcal{G}}$$
(45)

where a new tensor quantity $\hat{\mathcal{G}}$ is defined and its value can be eventually obtained (by using Eqs. (39), (41) and the relation $c = \pi \frac{N}{S} a^2 e$) in the form

$$\hat{\mathcal{G}} = \pi a^2 \frac{N}{S} \begin{pmatrix} P & 0\\ 0 & 1 - P \end{pmatrix} \frac{\hat{\sigma}}{\sqrt{\det \hat{\sigma}}}.$$
(46)

The exact limiting value (for $e \to 0$ or $c \to 0$) of the conductivity tensor derives from equation (44)

$$\hat{\sigma}_{eff} = \hat{\sigma} \left\{ \hat{I} - \hat{\mathcal{G}} \left[\hat{I} + \hat{\mathcal{G}} \right]^{-1} \right\}.$$
(47)

This ends the outline of the procedure to follow for a specific statistical distribution of cracks (for a given value of P).

4.2 Iterative and differential methods

We remark that the above equation is valid only for small crack density. Nevertheless, equations (46) and (47) are very useful to exploit the iterated homogenization method, that allows us to generalize our results to larger crack density values. Let us suppose that the effective conductivity tensor of a multi-cracked medium (containing an initial number N of cracks) is known to be $\hat{\sigma}_{eff}(N)$. If a small additional number ΔN of cracks is added to the matrix, then the change in the conductivity is approximately the same as if they were added to a uniform and homogeneous matrix characterized by $\hat{\sigma}_{eff}(N)$. The resulting conductivity will be therefore $\hat{\sigma}_{eff}(N + \Delta N)$, as obtained by equations (46) and (47), through the replacements: $\hat{\sigma} \rightarrow \hat{\sigma}_{eff}(N)$, $\hat{\sigma}_{eff} \rightarrow \hat{\sigma}_{eff}(N + \Delta N)$ and $N \rightarrow \Delta N$, i.e.

$$\hat{\sigma}_{eff}(N + \Delta N) = \hat{\sigma}_{eff}(N) \left\{ \hat{I} - \hat{\mathcal{G}} \left[\hat{I} + \hat{\mathcal{G}} \right]^{-1} \right\} \quad (48)$$

where

$$\hat{\mathcal{G}} = \pi a^2 \frac{\Delta N}{S} \begin{pmatrix} P & 0\\ 0 & 1 - P \end{pmatrix} \frac{\hat{\sigma}_{eff}(N)}{\sqrt{\det \hat{\sigma}_{eff}(N)}}.$$
 (49)

We obtain a set of difference equations that, in the limit of vanishingly small ΔN , converges to a set of nonlinear differential equations

$$\frac{d\hat{\sigma}_{eff}}{dN} = -\frac{\pi a^2}{S} \frac{\hat{\sigma}_{eff} \begin{pmatrix} P & 0\\ 0 & 1-P \end{pmatrix} \hat{\sigma}_{eff}}{\sqrt{\det \hat{\sigma}_{eff}}}$$
(50)

with the initial condition $\hat{\sigma}_{eff}(0) = \hat{\sigma}$. If the matrix conductivity tensor is $\hat{\sigma} = \text{diag}(\sigma_{10}, \sigma_{20})$ then the effective conductivity tensor will be of the similar form $\hat{\sigma}_{eff} = \text{diag}(\sigma_1, \sigma_2)$, and we obtain a couple of nonlinear differential equations

$$\frac{d\sigma_1}{dN} = -\frac{\pi a^2}{S} P \frac{\sigma_1^2}{\sqrt{\sigma_1 \sigma_2}} \tag{51}$$

$$\frac{d\sigma_2}{dN} = -\frac{\pi a^2}{S} (1-P) \frac{\sigma_2^2}{\sqrt{\sigma_1 \sigma_2}} \tag{52}$$

with $\sigma_1(0) = \sigma_{10}$ and $\sigma_2(0) = \sigma_{20}$. This differential problem completely describe the degradation process of an anisotropic conductor induced by an arbitrary orientational distribution of cracks. Hence, it takes into account both the original conductivity tensor of the matrix and the order parameter identifying the kind of random dispersion of defects. It is important to remark that the application of the iterative or differential technique to the multi-cracked conductor is possible only through the knowledge of the exact response of one single defect in an anisotropic environment (given by Eqs. (25) and (32)); in fact, when a given crack is added to a given existing distribution of cracks, it feels an anisotropic environment generated by the not uniform (for example parallel cracks or crack nearly aligned in a given direction) random orientation of the cracks themselves. It happens even if the original not cracked material is perfectly isotropic. We also prove that this differential problem is able to explain the differences of the degradation behaviour between the cases with parallel cracks (perfect order) and random orientation of cracks (complete disorder). In fact, as outlined in Appendix \mathbf{B} , it is possible to obtain the exact solution of equations (51) and (52) in closed form for an arbitrary Cauchy initial condition $\sigma_1(0) = \sigma_{10}$ and $\sigma_2(0) = \sigma_{20}$ (corresponding to the anisotropic matrix conductive behaviour). The final expressions can be eventually written

$$\sigma_{1} = \sigma_{10} \frac{4 \frac{1-P}{\left(1+\frac{\sigma_{10}}{\sigma_{20}}\right)P-1} \sqrt{\frac{\sigma_{10}}{\sigma_{20}} - \sqrt{\frac{1-P}{P}}}}{\left[1 - \frac{\sqrt{\frac{\sigma_{10}}{\sigma_{20}}} - \sqrt{\frac{1-P}{P}}}{\sqrt{\frac{\sigma_{10}}{\sigma_{20}}} + \sqrt{\frac{1-P}{P}}} e^{-\pi a^{2} \frac{N}{S}} \sqrt{P(1-P)}\right]^{2}}$$

$$\sigma_{2} = \sigma_{20} \frac{4 \frac{P}{\frac{\sigma_{20}}{\sigma_{10}} - P\left(1+\frac{\sigma_{20}}{\sigma_{10}}\right)}}{\sqrt{\frac{\sigma_{20}}{\sigma_{10}}} + \sqrt{\frac{1-P}{P}}} e^{-\pi a^{2} \frac{N}{S}} \sqrt{P(1-P)}} }{\left[1 - \frac{\sqrt{\frac{\sigma_{20}}{\sigma_{10}}} - \sqrt{\frac{P}{1-P}}}{\sqrt{\frac{\sigma_{20}}{\sigma_{10}}} + \sqrt{\frac{1-P}{1-P}}} e^{-\pi a^{2} \frac{N}{S}} \sqrt{P(1-P)}} \right]^{2} \cdot \left[1 - \frac{\sqrt{\frac{\sigma_{20}}{\sigma_{10}}} - \sqrt{\frac{P}{1-P}}}{\sqrt{\frac{\sigma_{20}}{\sigma_{10}}} + \sqrt{\frac{P}{1-P}}} e^{-\pi a^{2} \frac{N}{S}} \sqrt{P(1-P)}}\right]^{2} \cdot (54)$$

In the special case with an initially isotropic matrix having $\sigma_{10} = \sigma_{20} = \sigma_0$ we obtain the simpler solution

$$\sigma_{1} = \sigma_{0} \frac{4\frac{1-P}{2P-1} \frac{P-\sqrt{P(1-P)}}{P+\sqrt{P(1-P)}} e^{-\pi a^{2} \frac{N}{S} \sqrt{P(1-P)}}}{\left[1 - \frac{P-\sqrt{P(1-P)}}{P+\sqrt{P(1-P)}} e^{-\pi a^{2} \frac{N}{S} \sqrt{P(1-P)}}\right]^{2}}$$
(55)
$$\sigma_{2} = \sigma_{0} \frac{4\frac{P}{1-2P} \frac{(1-P)-\sqrt{P(1-P)}}{(1-P)+\sqrt{P(1-P)}} e^{-\pi a^{2} \frac{N}{S} \sqrt{P(1-P)}}}{\left[1 - \frac{(1-P)-\sqrt{P(1-P)}}{(1-P)+\sqrt{P(1-P)}} e^{-\pi a^{2} \frac{N}{S} \sqrt{P(1-P)}}\right]^{2}}$$
(56)

By means of these complete solutions it is possible to obtain the behaviour of several limiting cases as reported in Table 1. In particular we observe that the degradation of the conductivity with a parallel dispersion of cracks (in the direction orthogonal to the cracks) follows a power law of the form $\log \sigma_{1,2} \sim -\log N$ or, equivalently, $\sigma_{1,2} \sim 1/N^2$. On other hand, the disordered case $(P = \frac{\sigma_{20}}{\sigma_{10} + \sigma_{20}})$ if $\sigma_{10} \neq \sigma_{20}$ or P = 1/2 if $\sigma_{10} = \sigma_{20}$)

Table 1. Limiting cases obtained from equations (53)–(56) for specific values of the order parameters P. It is important to observe that the cases of perfect order (P = 0, 1) lead, for large N, to the power law $\log \sigma_{1,2} \sim -\log N$ while the disordered case $(P = \frac{\sigma_{20}}{\sigma_{10} + \sigma_{20}}, 1/2)$ leads to the exponential law $\log \sigma_{1,2} \sim -N$.

| Order parameter (P) | Isotropic matrix $(\sigma_{10} = \sigma_{20} = \sigma_0)$ | Anisotropic matrix $(\sigma_{10} \neq \sigma_{20})$ |
|--|--|--|
| P = 0 | $\begin{cases} \sigma_1 = \sigma_0 \\ \sigma_2 = \frac{\sigma_0}{\left(1 + \frac{1}{2}\pi a^2 \frac{N}{S}\right)^2} \end{cases}$ | $\begin{cases} \sigma_1 = \sigma_{10} \\ \sigma_2 = \frac{\sigma_{20}}{\left(1 + \frac{1}{2}\pi a^2 \frac{N}{S} \sqrt{\frac{\sigma_{20}}{\sigma_{10}}}\right)^2} \end{cases}$ |
| $P = \frac{\sigma_{20}}{\sigma_{10} + \sigma_{20}}$ $\left(P = \frac{1}{2} \text{ if } \sigma_{10} = \sigma_{20}\right)$ | $\begin{cases} \sigma_1 = \sigma_0 e^{-\frac{1}{2}\pi a^2 \frac{N}{S}} \\ \sigma_2 = \sigma_0 e^{-\frac{1}{2}\pi a^2 \frac{N}{S}} \end{cases}$ | $\begin{cases} \sigma_1 = \sigma_{10} e^{-\pi a^2 \frac{N}{S} \frac{\sqrt{\sigma_{10} \sigma_{20}}}{\sigma_{10} + \sigma_{20}}} \\ \sigma_2 = \sigma_{20} e^{-\pi a^2 \frac{N}{S} \frac{\sqrt{\sigma_{10} \sigma_{20}}}{\sigma_{10} + \sigma_{20}}} \end{cases}$ |
| P = 1 | $\begin{cases} \sigma_1 = \frac{\sigma_0}{\left(1 + \frac{1}{2}\pi a^2 \frac{N}{S}\right)^2} \\ \sigma_2 = \sigma_0 \end{cases}$ | $\begin{cases} \sigma_1 = \frac{\sigma_{10}}{\left(1 + \frac{1}{2}\pi a^2 \frac{N}{S} \sqrt{\frac{\sigma_{10}}{\sigma_{20}}}\right)^2} \\ \sigma_2 = \sigma_{20} \end{cases}$ |



Fig. 4. (Color online) Plot of $\log_{10} \frac{\sigma_1}{\sigma_{10}}$ in terms of $\log_{10} \pi a^2 \frac{N}{S}$ (left panel) and $\pi a^2 \frac{N}{S}$ (right panel) for the case with $\sigma_{10} = \sigma_{20} = \sigma_0$ (isotropic matrix). Different lines are parametrized by an increasing order parameter 0 < P < 1. The case with P = 0 (horizontal order, dotted-dashed lines) corresponds to the horizontal lines $\sigma_1 = \sigma_0$ (both in right and left panels); the case with P = 1/2 (solid lines) leads to an exponential behaviour described by $\log \sigma_1 \sim -N$ (straight line on the right panel); finally, the case with P = 1 (vertical order, dashed lines) corresponds to power law $\log \sigma_1 \sim -\log N$ represented by an asymptotic straight line on the left panel.

leads to the exponential law $\log \sigma_{1,2} \sim -N$ or, equivalently, $\sigma_{1,2} \sim \exp(-N/N_0)$ with a given N_0 (see Tab. 1 for details). Hence, the present model based on an arbitrary anisotropic environment, is able to reproduce the different degradation behaviours corresponding to the different orientational distributions [22–25].

One can find the explanation of this point in Figure 4 where we have considered a system with $\sigma_{10} = \sigma_{20} = \sigma_0$ (isotropic matrix). We have reported only the results related to σ_1 since those for σ_2 are simply symmetrical. The case with P = 1/2 (solid lines) leads to an exponential behaviour represented by a straight line on the right panel with bi-logarithmic scale; on the other hand, the case with P = 1 (dashed lines) corresponds to a power law represented by an asymptotic straight line on the left panel with semi-logarithmic scale.

Moreover, in Figure 5 we present the results for a strongly anisotropic environment characterized by $\sigma_{10} = 1$ and $\sigma_{20} = 100$ (in arbitrary units). One can find the three-dimensional plots of σ_1 (left panel) and σ_2 (right

panel) versus the order parameter P and the crack density $\pi a^2 \frac{N}{S}$. In the left panel we can observe that a strong anisotropy of the matrix amplifies the different behaviour of the degradation process with different values of the order parameter P. In particular it is evident that when P = 1 (vertical order) the power law degradation is much slower than the other cases with an exponential trend. Finally, in Figure 6 we describe the overall anisotropy of the multi-cracked conductor through the anisotropy ratio $\left|\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\right|$. Its value has been calculated for the case with $\sigma_{10} = 1$ and $\sigma_{20} = 100$ (in arbitrary units) and has been shown versus the order parameter P and the crack density $\pi a^2 \frac{N}{S}$. It is evident a curved line on the plane $(P, \pi a^2 \frac{N}{S})$ where the anisotropy ratio is zero and where the system is therefore equivalent to an isotropic one. It is simple to verify that, when $\sigma_{20} > \sigma_{10}$, the starting point of the curve for P = 0 occurs at $\pi a^2 \frac{N}{S} = 2\left(1 - \sqrt{\sigma_{10}/\sigma_{20}}\right)$, i.e. at $\pi a^2 \frac{N}{S} = 9/5$ in our case with $\sigma_{10} = 1$ and $\sigma_{20} = 100$. Such a point can be easily identified in Figure 6. Moreover,



Fig. 5. (Color online) Three-dimensional plots of σ_1 (left panel) and σ_2 (right panel) for a strongly anisotropic system with $\sigma_{10} = 1$ and $\sigma_{20} = 100$ (in arbitrary units). The effective conductivities are shown versus the order parameter P and the crack density $\pi a^2 \frac{N}{S}$.



Fig. 6. (Color online) Plot of the anisotropy ratio $\left|\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\right|$, measuring the degree of anisotropy of the overall system. It has been calculated for the case with $\sigma_{10} = 1$ and $\sigma_{20} = 100$ (in arbitrary units) and it has been shown versus the order parameter P and the crack density $\pi a^2 \frac{N}{S}$. It is evident a curved line on the plane $(P, \pi a^2 \frac{N}{S})$ where the anisotropy ratio is zero and where the system is therefore equivalent to an isotropic one.

it is also possible to prove that, when $\pi a^2 \frac{N}{S} \to \infty$, then the curve approaches the value P = 1/2, as represented in Figure 6.

Other relevant applications of iterative (or incremental) and differential schemes are available in literature for a variety of heterogeneous structures. The incremental method converges to the differential one (for a large number of infinitesimal steps) both in our development and in reference [70] dealing with bi-anisotropic composites. As expected, the two approaches may lead to different results if we consider a finite number of discrete incremental steps within the iterative scheme [70]. Interestingly, in reference [71] the incremental formalism is also compared with the symmetric Bruggeman formalism. There, it has been proved that if there is a strong contrast between the components the two predictions differ significantly. We will describe an application of this method in the next Section in order to predict a percolative behaviour.

Finally, it is important to draw some comparisons between equations (55) and (56) and the following ones previously obtained in reference [72]

$$\sigma_1 = \sigma_0 e^{-\pi a^2 \frac{N}{S}P} \tag{57}$$

$$\sigma_2 = \sigma_0 e^{-\pi a^2 \frac{N}{S}(1-P)}.$$
 (58)

In reference [72] the solutions are approximated and they do not take into account the anisotropy of the system: in other words, in reference [72] we have neglected the fact that when a crack is added to a given population (through the iterative process), it feels an anisotropic effective matrix because of the pseudo-random orientation described by P. On the contrary, in the present investigation the anisotropic character induced by the not uniform random orientation of the cracks is properly taken into account. This is the central point which leads to the correct degradation behaviour: in particular, the real anisotropic character is essential for obtaining the power law decay with parallel cracks [24,25]. In fact, equations (55)and (56) are identical to equations (57) and (58) for P = 1/2 (without anisotropic effects) and the larger deviation is obtained for σ_1 when P = 1 and for σ_2 when P = 0 (with the largest anisotropic effects). In Figure 7 one can find the relative percent error e[%] for σ_1 , obtained comparing equations (55) and (57). The error is always zero for P = 0 and P = 1/2 and is very high with large crack density and for P = 1. In fact, equation (57) is unable to reproduce the expected power law decay of σ_1 for the population of vertical cracks.

4.3 Percolative behaviour

We have described in the previous Section an exponential degradation of the conductivity for a uniform distribution of cracks on the plane. For an isotropic matrix we have obtained $\sigma_{eff} = \sigma_1 = \sigma_2 = \sigma_0 e^{-\frac{1}{2}\pi a^2 \frac{N}{S}}$. Therefore, the limit



Fig. 7. (Color online) Plot of the relative percent error e[%] for σ_1 versus the order parameter P and the crack density $\pi a^2 \frac{N}{S}$, obtained comparing equations (55) and (57). The error is always zero for P = 0 and P = 1/2 and is very high with large crack density and for P = 1.

 $\sigma_{eff} = 0$ is only reached for $\alpha = \pi a^2 \frac{N}{S} \to \infty$, which means that this theory does not predict percolation. However, in reality percolation occurs for $\alpha = \pi a^2 \frac{N}{S} = 4.49$ [25,73]; at this crack density the effective conductivity vanishes. We propose in this Section a simple explication of this behaviour by means of the symmetric Bruggeman formalism [74,75]. To begin, we take into consideration a simple initial situation which is useful for the following derivation. We consider a strongly dilute dispersion of random oriented isotropic elliptic particles (aspect ratio e and conductivity σ_i) embedded in an isotropic matrix σ_0 with volume fraction c. In this case the results of the previous sections lead to the effective conductivity $\sigma_{eff} = F(\sigma_0, \sigma_i, c)$ given by the following expression

$$F(\sigma_0, \sigma_i, c) = \sigma_0 + \frac{1}{2}c(\sigma_i - \sigma_0)$$

$$\times \left[\frac{\sigma_0}{\sigma_0 + \frac{e}{e+1}(\sigma_i - \sigma_0)} + \frac{\sigma_0}{\sigma_0 + \frac{1}{e+1}(\sigma_i - \sigma_0)} \right].$$
(59)

The symmetric Bruggeman formalism is typically applied to a mixture of grains (having completely haphazard distribution of size, position, and shape) made by two (or more) different media. We consider grained materials formed by two different phases having conductivities σ_A and σ_B , and volume fractions $c_A = 1 - c$ and $c_B = c$, respectively. We suppose that the mixing rule for this system is given by $\sigma_{eff} = G(\sigma_A, \sigma_B, c)$. We will assume that the only structural information about the mixture are c and the predominant aspect ratios e_A and e_B for the grains of the two phases. We remark that in this problem we may not distinguish between a "matrix" and the "inhomogeneities": accordingly, the concept of "embeddeding" so usefully developed in the previous sections is no longer relevant. Nevertheless, it is possible to use equation (59)for the function F in order to find some properties of a random mixture, i.e. of the function G. To this aim, let us examine the situation of the random mixture for very low values of c: we may think that a low value of c is reached

when the structure contains only very small elements of the second medium dispersed in the matrix of the first medium. In this limiting case, equation (59) (obtained for diluted dispersion of particles) is still valid: so, we may write down the derivative of G with respect to the volume fraction c (in the limit of vanishingly small c)

$$\frac{\partial G}{\partial c} \left(\sigma_A, \sigma_B, c = 0 \right) = \frac{1}{2} \left(\sigma_B - \sigma_A \right)$$

$$\times \left[\frac{\sigma_A}{\sigma_A + \frac{e_B}{e_B + 1} \left(\sigma_B - \sigma_A \right)} + \frac{\sigma_A}{\sigma_A + \frac{1}{e_B + 1} \left(\sigma_B - \sigma_A \right)} \right].$$
(60)

By using this expression we may solve the complete problem of a linear random mixture composed by different homogeneous components randomly mixed together [74,75]. In fact, we may think to add to the overall system (with conductivity $\sigma_{eff} = G(\sigma_A, \sigma_B, c)$ where c is now arbitrary) another grain with volume $\Delta V \ll V$ and conductivity σ_A . The conductivity of the resulting new mixture can be described by homogenization

$$\sigma_{eff}^{A} = G\left(\sigma_{eff}, \sigma_{A}, \frac{\Delta V}{\Delta V + V}\right) \tag{61}$$

where we have considered a two-phase mixture between σ_{eff} (the original medium) and σ_A (the added grain). Since $\Delta V \ll V$, we can expand G up to the first order in $\Delta V/(\Delta V + V)$ and therefore we have

$$\sigma_{eff}^{A} = \sigma_{eff} + \frac{\partial G}{\partial c} \left(\sigma_{eff}, \sigma_{A}, 0 \right) \frac{\Delta V}{V}.$$
 (62)

The same result can be obtained by adding a grain having conductivity σ_B to the original grained material

$$\sigma_{eff}^{B} = \sigma_{eff} + \frac{\partial G}{\partial c} \left(\sigma_{eff}, \sigma_{B}, 0 \right) \frac{\Delta V}{V}.$$
 (63)

At the end of the procedure the unknown function G is obtained in implicit version

$$\sigma_{eff} = (1 - c)\sigma_{eff}^A + c\sigma_{eff}^B \tag{64}$$

where we have given the probability 1-c to the insertion of the grain σ_A and the probability c to the insertion of the grain σ_B . By simplifying the previous expression we have

$$0 = (1 - c)\frac{\partial G}{\partial c} \left(\sigma_{eff}, \sigma_A, 0\right) + c\frac{\partial G}{\partial c} \left(\sigma_{eff}, \sigma_B, 0\right) \quad (65)$$

which implies, by using equation (60), the most important result of the effective medium theory, namely

$$0 = (1 - c) \left[\frac{1}{\frac{\sigma_{eff}}{\sigma_A - \sigma_{eff}} + \frac{e_A}{e_A + 1}} + \frac{1}{\frac{\sigma_{eff}}{\sigma_A - \sigma_{eff}} + \frac{1}{e_A + 1}} \right] \quad (66)$$
$$+ c \left[\frac{1}{\frac{\sigma_{eff}}{\sigma_B - \sigma_{eff}} + \frac{e_B}{e_B + 1}} + \frac{1}{\frac{\sigma_{eff}}{\sigma_B - \sigma_{eff}} + \frac{1}{e_B + 1}} \right].$$

The solution σ_{eff} of the previous equation represents the value assumed by the function $\sigma_{eff} = G(\sigma_A, \sigma_B, c)$. Now,

$$\sigma_{eff} = \begin{cases} \sigma_0 \frac{4-\alpha}{4+\alpha} & \text{if } \alpha \le 4\\ 0 & \text{if } \alpha > 4 \end{cases}$$
(67)

where $\alpha = \pi a^2 \frac{N}{S}$. This theory has predicted a percolation threshold corresponding to $\alpha = 4$, which shows a quite good agreement with the value reported in literature $\alpha = 4.49$ [25,73]. We observe that, while the iterative and differential methods are able to capture the degradation trends for different orientational distributions, they are not able to describe the percolative scenario occurring beyond the percolation threshold. Nevertheless, as described in the present section, the symmetric Bruggeman formalism predicts quite correctly the value of such a percolation threshold, at least for randomly oriented cracks.

5 Conclusions

In this paper we have analysed the conduction properties of two-dimensional heterogeneous multi-cracked structures. In the first part of the work we have obtained a result concerning an anisotropic single inhomogeneity (of elliptic shape) embedded in an anisotropic host matrix. It has been obtained through the Green function for the two-dimensional anisotropic electrostatics applied to the concepts of eigenfield and inclusion. This result can be written in a simple algebraic form allowing for direct applications to paradigmatic heterogeneous materials. In fact, in the second part of this work we have studied the conduction properties of multi-cracked conductors. An initial homogenization theory has been developed for dealing with a dilute dispersion of cracks; then, it has been generalized to higher crack densities through an iterative scheme which leads to a system of nonlinear differential equations. Such an approach takes into consideration the interactions among cracks in an implicit but efficient way. We have indeed proved that the model is able to explain the degradation process of a multi-cracked material for different orientational distribution of the cracks within the host matrix. In particular it predicts an exponential law for randomly oriented cracks and a power law for parallel cracks, as observed in recent literature. In the case of randomly oriented cracks we have also presented a simple technique to evaluate the percolation threshold. A future investigation will be devoted to the applications of these methodologies not only to the transport properties but also to the elastic response of damaged materials.

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Appendix A: An integral calculation

We want to determine the value of the integral

$$\hat{\mathcal{I}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\boldsymbol{n} \otimes \boldsymbol{n}}{(\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}) (\boldsymbol{n} \cdot \hat{\eta} \boldsymbol{n})} d\alpha \qquad (A.1)$$

for two arbitrary symmetric and positive definite tensors $\hat{\sigma}$ and $\hat{\eta}$. We convert the real integral on the range $(0, 2\pi)$ to a complex one over the unit circle through the change of variable $z = e^{i\alpha}$. Since $\boldsymbol{n} = (\cos \alpha, \sin \alpha)$ with $\cos \alpha = (1/2)(z+1/z)$ and $\sin \alpha = (1/2i)(z-1/z)$ we have

$$\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n} = \frac{1}{4z^2} \mathcal{D}_{\sigma}, \quad \boldsymbol{n} \cdot \hat{\eta} \boldsymbol{n} = \frac{1}{4z^2} \mathcal{D}_{\eta}$$
(A.2)

where

$$\mathcal{D}_{\sigma} = (z^{2}+1)^{2} \sigma_{11} - 2i\sigma_{12} (z^{4}-1) - (z^{2}-1)^{2} \sigma_{22}$$
(A.3)

$$\mathcal{D}_{\eta} = (z^{2}+1)^{2} \eta_{11} - 2i\eta_{12} (z^{4}-1) - (z^{2}-1)^{2} \eta_{22}.$$
(A.4)

Moreover, the numerator in the integrand can be written as

$$\boldsymbol{n} \otimes \boldsymbol{n} = \frac{1}{4z^2} \hat{\mathcal{N}}(z)$$
 (A.5)

where

$$\hat{\mathcal{N}}(z) = \begin{pmatrix} z^4 + 2z^2 + 1 & i - iz^4 \\ i - iz^4 & -z^4 + 2z^2 - 1 \end{pmatrix}.$$
 (A.6)

Hence, since $d\alpha = dz/(iz)$, we obtain

$$\hat{\mathcal{I}} = \frac{2}{i\pi} \int_{\partial D} \frac{\mathcal{N}(z)zdz}{\mathcal{D}_{\sigma}(z)\mathcal{D}_{\eta}(z)}$$
(A.7)

where ∂D is the unit circle on the complex plane. In order to apply the Cauchy residue theorem we start by searching for the poles of the integrand. We find that $\mathcal{D}_{\sigma}(z) = 0$ if

$$z^{2} = \frac{-\mathrm{tr}\hat{\sigma} \pm 2\sqrt{\det\hat{\sigma}}}{\sigma_{11} - \sigma_{22} - 2i\sigma_{12}}$$
(A.8)

whence, by extracting the square root, we can calculate four zeros for the polynomial $\mathcal{D}_{\sigma}(z)$. From equation (A.8) we directly obtain

$$|z^{2}| = \sqrt{\frac{\mathrm{tr}\hat{\sigma} \mp 2\sqrt{\mathrm{det}\,\hat{\sigma}}}{\mathrm{tr}\hat{\sigma} \pm 2\sqrt{\mathrm{det}\,\hat{\sigma}}}} \tag{A.9}$$

and, therefore, the poles within the unit circle |z| < 1 are two and, more precisely, those corresponding to the plus sign in equation (A.8). Such two poles, from now on, will be indicated as $z_{\sigma 1}$ and $z_{\sigma 2}$

$$z_{\sigma 1,2} = \pm \sqrt{\frac{-\text{tr}\hat{\sigma} + 2\sqrt{\det\hat{\sigma}}}{\sigma_{11} - \sigma_{22} - 2i\sigma_{12}}}.$$
 (A.10)

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Similarly, other two poles within the unit circle come from the polynomial $\mathcal{D}_{\eta}(z)$ and they will be named $z_{\eta 1}$ and $z_{\eta 2}$

$$z_{\eta 1,2} = \pm \sqrt{\frac{-\mathrm{tr}\hat{\eta} + 2\sqrt{\mathrm{det}\,\hat{\eta}}}{\eta_{11} - \eta_{22} - 2i\eta_{12}}}.$$
 (A.11)

The application of the residue theorem leads immediately to

$$\hat{\mathcal{I}} = 4 \sum_{j=1}^{2} \operatorname{Res} \left[\frac{\hat{\mathcal{N}}(z)z}{\mathcal{D}_{\sigma}(z)\mathcal{D}_{\eta}(z)}, z_{\sigma j} \right] \\ + 4 \sum_{j=1}^{2} \operatorname{Res} \left[\frac{\hat{\mathcal{N}}(z)z}{\mathcal{D}_{\sigma}(z)\mathcal{D}_{\eta}(z)}, z_{\eta j} \right]$$
(A.12)

$$=4\sum_{j=1}^{2}\lim_{z\to z_{\sigma j}}\frac{z(z-z_{\sigma j})\hat{\mathcal{N}}(z)}{\mathcal{D}_{\sigma}(z)\mathcal{D}_{\eta}(z)}$$
$$+4\sum_{j=1}^{2}\lim_{z\to z_{\eta j}}\frac{z(z-z_{\eta j})\hat{\mathcal{N}}(z)}{\mathcal{D}_{\sigma}(z)\mathcal{D}_{\eta}(z)}.$$
(A.13)

By a simple direct evaluation we obtain

$$\lim_{z \to z_{\sigma j}} \frac{\mathcal{D}_{\sigma}(z)}{z - z_{\sigma j}} = 8z_{\sigma j}\sqrt{\det \hat{\sigma}}, \quad j = 1, 2$$
(A.14)

$$\lim_{z \to z_{\eta j}} \frac{\mathcal{D}_{\eta}(z)}{z - z_{\eta j}} = 8 z_{\eta j} \sqrt{\det \hat{\eta}}, \ j = 1, 2.$$
 (A.15)

Combining these results with equation (A.13) we obtain a first form for the integral value

$$\hat{\mathcal{I}} = \frac{1}{2} \sum_{j=1}^{2} \frac{\hat{\mathcal{N}}(z_{\sigma j})}{\sqrt{\det \hat{\sigma}} \mathcal{D}_{\eta}(z_{\sigma j})} + \frac{1}{2} \sum_{j=1}^{2} \frac{\hat{\mathcal{N}}(z_{\eta j})}{\sqrt{\det \hat{\eta}} \mathcal{D}_{\sigma}(z_{\eta j})}.$$
(A.16)

Now, since $\hat{\mathcal{N}}(z)$, $\mathcal{D}_{\sigma}(z)$ and $\mathcal{D}_{\eta}(z)$ directly depend on z^2 , the first couple of terms (j = 1, 2) assume the same value when calculated for $z_{\sigma 1}$ or $z_{\sigma 2}$. The same thing happens for the second couple of terms and, therefore, we may simplify equation (A.16) to

$$\hat{\mathcal{I}} = \frac{\hat{\mathcal{N}}(z_{\sigma 1})}{\sqrt{\det \hat{\sigma}} \mathcal{D}_{\eta}(z_{\sigma 1})} + \frac{\hat{\mathcal{N}}(z_{\eta 1})}{\sqrt{\det \hat{\eta}} \mathcal{D}_{\sigma}(z_{\eta 1})}.$$
 (A.17)

To conclude, a very long but straightforward substitution of equations (A.10) and (A.11) in equation (A.17) leads to the result stated in equation (22). Interestingly enough, if $\hat{\eta} = \hat{I}$ or $\hat{\eta} = \hat{\sigma}$ we have the special results

$$\frac{1}{2\pi} \int_{0}^{2\pi} \frac{\boldsymbol{n} \otimes \boldsymbol{n}}{\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}} d\alpha = \frac{\hat{I} + \hat{\sigma}^{-1} \sqrt{\det \hat{\sigma}}}{\operatorname{tr} \hat{\sigma} + 2\sqrt{\det \hat{\sigma}}} \quad (A.18)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\boldsymbol{n} \otimes \boldsymbol{n}}{\left(\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}\right)^2} d\alpha = \frac{\hat{\sigma}^{-1}}{2\sqrt{\det \hat{\sigma}}}$$
(A.19)

and applying the trace operator we also have

$$\frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\alpha}{\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}} = \frac{1}{\sqrt{\det \hat{\sigma}}}$$
(A.20)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{\left(\boldsymbol{n} \cdot \hat{\sigma} \boldsymbol{n}\right)^2} = \frac{\mathrm{tr}\hat{\sigma}}{2 \det \hat{\sigma} \sqrt{\det \hat{\sigma}}}.$$
 (A.21)

Appendix B: Solution of the differential problem

We search the solution of equations (51) and (52) with the initial conditions $\sigma_1(0) = \sigma_{10}$ and $\sigma_2(0) = \sigma_{20}$. From the first equation we directly obtain

$$\sigma_2 = \left(\frac{\pi a^2}{S}\right)^2 \frac{P^2 \sigma_1^3}{\left(\frac{d\sigma_1}{dN}\right)^2} \tag{B.22}$$

and, substituting equation (B.22) into equation (52) we have a pure differential equation in the unknown σ_1

$$3\left(\frac{d\sigma_1}{dN}\right)^2 - 2\sigma_1 \frac{d^2\sigma_1}{dN^2} - \left(\frac{\pi a^2}{S}\right)^2 P\left(1 - P\right)\sigma_1^2 = 0.$$
(B.23)

This is a second order nonlinear differential equation of the form F(y, y', y'') = 0 (where y = y(x), y' = dy/dxand $y'' = d^2y/dx^2$) and, typically, it can be solved by the transformation y' = f(y) where the function f is the new unknown. In our case we therefore let $d\sigma_1/dN = f(\sigma_1)$ and we develop the second derivative as follows

$$\frac{d^2\sigma_1}{dN^2} = \frac{d}{dN}\frac{d\sigma_1}{dN} = \frac{d}{dN}f(\sigma_1) = \frac{df}{d\sigma_1}\frac{d\sigma_1}{dN} = f\frac{df}{d\sigma_1}.$$
(B.24)

By adopting this change of variable, equation (B.23) assumes the form of the following first order differential equation

$$\frac{df}{d\sigma_1} = -\frac{1}{2} \left(\frac{\pi a^2}{S}\right)^2 P(1-P)\frac{\sigma_1}{f} + \frac{3}{2}\frac{f}{\sigma_1}.$$
 (B.25)

This is an homogeneous equation of the type $dy/dx = \phi(y/x)$ and it can be straightforwardly approached through the introduction of u = y/x, where u(x) is to be determined. In our case it means that we define $u(\sigma_1) = f(\sigma_1)/\sigma_1$ from which we have $df/d\sigma_1 = \sigma_1 du/d\sigma_1 + u$. At last, equation (B.25) is converted to

$$2\sigma_1 \frac{du}{d\sigma_1} = -\left(\frac{\pi a^2}{S}\right)^2 P(1-P)\frac{1}{u} + u \qquad (B.26)$$

which is a separable equation leading to

$$\int \frac{2udu}{u^2 - \left(\frac{\pi a^2}{S}\right)^2 P(1-P)} = \int \frac{d\sigma_1}{\sigma_1} \qquad (B.27)$$

or, equivalently, to

$$\log|u^2 - \left(\frac{\pi a^2}{S}\right)^2 P(1-P)| = \log|\sigma_1| + C.$$
(B.28)

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By introducing the integration constant $K = \exp(C) > 0$ and we have

$$u = \pm \sqrt{K\sigma_1 + \left(\frac{\pi a^2}{S}\right)^2 P(1-P)}.$$
 (B.29)

By using the previous definition $u(\sigma_1) = f(\sigma_1)/\sigma_1$ we come back to the function $f(\sigma_1) = \sigma_1 u(\sigma_1)$. Consequently, we can write the final differential equation $d\sigma_1/dN =$ $f(\sigma_1)$: since $\sigma_1(N)$ must be a decreasing function of N(degradation process) we must choose the minus sign in the previous solution, by obtaining

$$\frac{d\sigma_1}{dN} = -\sigma_1 \sqrt{K\sigma_1 + \left(\frac{\pi a^2}{S}\right)^2 P(1-P)}.$$
 (B.30)

This equation, being again separable, can be integrated

$$\int_{\sigma_{01}}^{\sigma_1} \frac{d\sigma_1}{\sigma_1 \sqrt{K\sigma_1 + \left(\frac{\pi a^2}{S}\right)^2 P(1-P)}} = -N. \quad (B.31)$$

The integration constant K can be found from equation (51) written for N = 0

$$\frac{d\sigma_1}{dN}(N=0) = -\frac{\pi a^2}{S} P \sigma_{10}^{3/2} \sigma_{20}^{-1/2}$$
(B.32)

combined with equation (B.30), always written for N = 0; the result is

$$K = \left(\frac{\pi a^2}{S}\right)^2 P \frac{1}{\sigma_{10}} \left[\left(1 + \frac{\sigma_{10}}{\sigma_{20}}\right) P - 1 \right]. \quad (B.33)$$

To conclude, equation (B.31) can be simply developed by recalling that

$$\int \frac{dx}{x\sqrt{\alpha + Kx}} = \frac{1}{\sqrt{\alpha}} \log \left| \frac{\sqrt{\alpha + Kx} - \sqrt{\alpha}}{\sqrt{\alpha + Kx} + \sqrt{\alpha}} \right| + c \quad (B.34)$$

for $\alpha > 0$ (in our case we define $\alpha = (\pi a^2/S)^2 P(1-P)$) and for an arbitrary constant *c*. Therefore, equation (B.31) can be written as

$$\frac{\mathcal{Y} - \sqrt{P(1-P)}}{\mathcal{Y} + \sqrt{P(1-P)}} = \mathcal{X}$$
(B.35)

where

$$\mathcal{Y} = \sqrt{P \frac{\sigma_1}{\sigma_{10}} \left[\left(1 + \frac{\sigma_{10}}{\sigma_{20}} \right) P - 1 \right] + P(1 - P)$$
(B.36)

and

$$\mathcal{X} = \frac{P\sqrt{\frac{\sigma_{10}}{\sigma_{20}}} - \sqrt{P(1-P)}}{P\sqrt{\frac{\sigma_{10}}{\sigma_{20}}} + \sqrt{P(1-P)}} e^{-\pi a^2 \frac{N}{S}\sqrt{P(1-P)}}.$$
 (B.37)

equation (B.35) can be inverted resulting in

$$\mathcal{Y} = \frac{1 + \mathcal{X}}{1 - \mathcal{X}} \sqrt{P(1 - P)}$$
(B.38)

and
$$\sigma_1$$
 can be obtained from equation (B.36)

$$\sigma_{1} = \sigma_{10} \frac{\mathcal{Y}^{2} - P(1 - P)}{P\left[\left(1 + \frac{\sigma_{10}}{\sigma_{20}}\right)P - 1\right]}$$
$$= \sigma_{10} \frac{4\mathcal{X}}{\left(1 - \mathcal{X}\right)^{2}} \frac{1 - P}{\left(1 + \frac{\sigma_{10}}{\sigma_{20}}\right)P - 1}.$$
(B.39)

Finally, by substituting equation (B.37) in equation (B.39) we obtain the complete solution shown in equation (53). The second solution for σ_2 can be simply obtained by inverting the subscripts '1' and '2' and the symbols P and 1 - P in the first solution for σ_1 .

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