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Two-dimensional disordered lattice networks with substrate

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Abstract

The paper deals with the development of a theory for describing two-dimensional (2D) random lattice networks of resistors with a particular topology. We consider a 2D anisotropic random lattice where each node of the network is connected to a reference node (substrate) through a given random resistor. This topology is of great interest both for theoretical and practical applications. Moreover, the theory is able to take into account the similar, but more interesting problem with a capacitive coupling with the substrate. The analytical results allow us to obtain the average behaviour of such networks, i.e. the electrical characterisation of the corresponding physical systems. This effective medium theory is developed starting from the properties of the lattice Green's function of the network and from an ad hoc mean field procedure. An accurate analytical study of the related lattice Green's functions has been conducted obtaining closed-form results expressed in terms of elliptic integrals. All the theoretical results have been verified by means of numerical Monte-Carlo simulations obtaining a remarkably good agreement between numerical and theoretical values, both in resistive and capacitive systems.
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Keywords: Random networks; Green functions; Mean field theory

1. Introduction

Disordered resistors networks have been, for many years, very useful tools to model transport phenomena in heterogeneous or composite physical systems. The first studies, from a general point of view, were developed by Kirkpatrick in the context of the isotropic transport and the percolation in random lattice [1,2]. In these works, the theoretical description of conduction was provided by a so-called effective medium theory. This theory, originally formulated to describe the conductivity of binary mixture [3,4], has been extended and adapted to treat disordered networks. Moreover, some attempts to generalise the theory to anisotropic random networks were made to clarify some general aspects of conduction in anisotropic materials [5]. During the evolution of such theories many approaches have been used to obtain statistical information about the behaviour of heterogeneous systems. In Ref. [6] exact fields calculations lead to exact effective properties in some particular cases; moreover, alternative theoretical circuit theory approaches have been adopted to obtain

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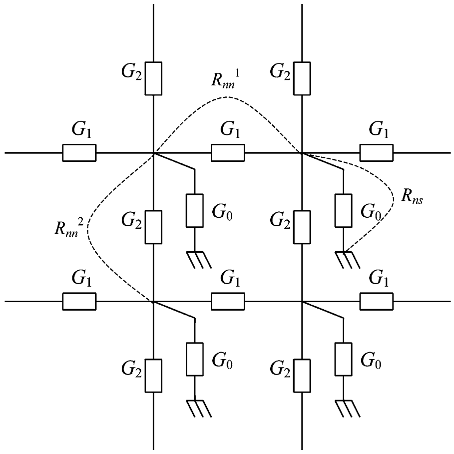

Fig. 1. 2D version of the general topology network used in this work. One can observe the anisotropy of the network and the definition of the characteristic node–node and node–substrate resistance.

the electrical properties of statistical mixtures [7]. More recently, the theories for linear random networks have been generalised to the case of non-linear random networks obtaining the equivalent non-linear behaviour of such heterogeneous systems [8,9]. Furthermore, an extremely refined series expansion has been developed for the macroscopic conductivity of a 3D random resistor network in terms of the contrast between the two kinds of conductors that appear in the network [10]. It was done by means of the Green function for the simple cubic lattice and by considering all the graphs contributing to each term of the series. This work has been generalised [11] to the case of the non-linear behaviour of a simple cubic, two-component, random resistor network. In the linear case the expansion has been worked out up to seventh order [10] and in the non-linear case it has been carried up to the third order [11].

In all these cases the most considered topology is the simple 2D or 3D grid of resistors, which mimics the heterogeneity in 2D or 3D composite materials. In this work, we devote our attention to a generalised topology where each node of the 2D grid is connected to another external node (called substrate in the following) by means of a given resistor (or impedance in the most general case). In Fig. 1 one can find an example of such a network. The random character of the network may regard the resistors in the grid or the resistors towards the substrate or in the more complex case both of them. Moreover, the statistical distribution of the resistor values in the grid may follow different probability laws in the different directions of the grid. This possibility allows us to describe anisotropy of the system. The method applied to develop the general theory is based on two main steps: firstly an analysis of the homogeneous networks with substrate based on the lattice Green's functions is performed. This approach [12,13] permits to obtain exact results about the electrical behaviour of infinite regular lattice networks. The second step consists in applying an ad hoc averaging procedure based on the effective medium theory. This approach is a generalisation of the standard one in order to take into account the electrical interaction with the substrate. The application of the two steps allows us to find a strong conceptual connection between the lattice Green's function of the network and the problem of obtaining the average behaviour of random grids. In other words, we may say that the lattice Green's functions introduced in this work are very useful tools to develop effective medium theories for general topology networks. An accurate analytical study of the related lattice Green's functions has been conducted obtaining many closed form results expressed in terms of elliptic integrals. All the theoretical results have been verified by means of numerical Monte-Carlo simulations obtaining a remarkably good agreement between numerical and theoretical values.

2. Lattice resistance functions

To approach the problem of the general disordered resistance networks it is important to know, as preliminary information, the resistive behaviour of homogeneous infinite lattice networks. We start by analysing the following general lattice topology: we take into consideration an anisotropic 2D homogeneous grid. It means that we define different values for the conductances aligned along the different lattice directions. Homogeneous grid means that all the conductances in a given direction have the same value. In particular the value of all the conductances in direction h ($h = 1, 2$) will be indicated with G_h. Moreover, each node of the grid is connected with a substrate node (another external node not belonging to the grid) through a conductance G_0. The homogeneous network with substrate is represented in Fig. 1. We represent the position of a given node with integer coordinates $\bar{x} \in Z^2$ and we consider the associated electrical potential, indicated as $V(\bar{x}) \forall \bar{x} \in Z^2$. To characterise such a network, we take into consideration two arbitrary nodes i and j (represented by the lattice positions $\bar{x}^i$ and $\bar{x}^j$) and the common node 0 of the substrate. We suppose that two given currents I_i and I_j flow in terminal connected to the nodes $\bar{x}^i$ and $\bar{x}^j$, in order to define a two-port network (see Fig. 2 for details). The current Kirchhoff law applied to the generic node $\bar{x}$ reads:

$$\sum_{k=1}^{2} G_k[2V(\bar{x}) - V(\bar{x} + \bar{e}_k) - V(\bar{x} - \bar{e}_k)] + G_0 V(\bar{x}) = I_i \delta(\bar{x}, \bar{x}^i) + I_j \delta(\bar{x}, \bar{x}^j), \tag{1}$$

where $\delta(\bar{x}, \bar{y})$ is the Kronecker's delta function ($\delta(\bar{x}, \bar{y}) = 1$ if $\bar{x} = \bar{y}$ and $\delta(\bar{x}, \bar{y}) = 0$ if $\bar{x} \neq \bar{y}$). We define the following Fourier transform ($\bar{k} \in \Re^2$):

$$\Im[V(\bar{x})] = f(\bar{k}) = \sum_{\bar{x} \in Z^2} V(\bar{x}) \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}}. \tag{2}$$

By using, in straightforward way, the following transformation rules:

$$\Im[V(\bar{x} + \bar{e}_h)] = f(\bar{k})\mathrm{e}^{\mathrm{i}k_h}; \quad \Im[V(\bar{x} - \bar{e}_h)] = f(\bar{k})\mathrm{e}^{-\mathrm{i}k_h}; \quad \Im[\delta(\bar{x}, \bar{x}^s)] = \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}^s} \tag{3}$$

we obtain the explicit solution of Eq. (1) in the transformed domain:

$$f(\bar{k}) = \frac{I_i \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}^i} + I_j \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}^j}}{\Sigma_{h=1}^{2} G_h \left[2 - \mathrm{e}^{\mathrm{i}k_h} - \mathrm{e}^{-\mathrm{i}k_h}\right] + G_0}. \tag{4}$$

The general expression for the inverse transform is given by

$$V(\bar{x}) = \Im^{-1}[f(\bar{k})] = \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} f(\bar{k}) \mathrm{e}^{\mathrm{i}\bar{k}\cdot\bar{x}} \, \mathrm{d}\bar{k}. \tag{5}$$

Therefore, substituting Eq. (4) into Eq. (5) we obtain, the following integral expression for the electrical potential in an arbitrary node of the lattice network:

$$V(\bar{x}) = \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \frac{I_i \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}^i} + I_j \mathrm{e}^{-\mathrm{i}\bar{k}\cdot\bar{x}^j}}{\sum_{h=1}^{2} G_h \left[2 - \mathrm{e}^{\mathrm{i}k_h} - \mathrm{e}^{-\mathrm{i}k_h}\right] + G_0} \mathrm{e}^{\mathrm{i}\bar{k}\cdot\bar{x}} \, \mathrm{d}\bar{k}. \tag{6}$$

This expression may be applied to the nodes of interest, by defining the potentials of the nodes $\bar{x}^i$ and $\bar{x}^j$, where the current generators have been connected. The potentials in these points are linearly related to the two current I_i and I_j by means of the impedance matrix $\tilde{Z}$ [14]:

$$\begin{cases} V_i = V(\bar{x}_i) \\ V_j = V(\bar{x}_j) \end{cases} \Rightarrow \begin{cases} V_i = \tilde{Z}_{ii} I_i + \tilde{Z}_{ij} I_j, \\ V_j = \tilde{Z}_{ji} I_i + \tilde{Z}_{jj} I_j. \end{cases} \tag{7}$$

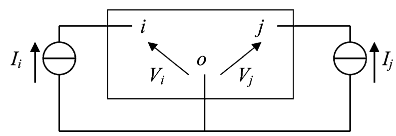

Fig. 2. Scheme of the lattice network where two arbitrary nodes and the substrate node are considered and connected to the current generators described in the main text.

By drawing a comparison between Eqs. (6) and (7) we may find out the explicit expressions for the impedance matrix elements:

$$\begin{cases} \tilde{Z}_{ii} = \tilde{Z}_{jj} = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}\frac{1}{\sum_{h=1}^{2}G_h\left[2-e^{ik_h}-e^{-ik_h}\right]+G_0}\,d\bar{k}, \\ \tilde{Z}_{ij} = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}\frac{e^{i\bar{k}\cdot(\bar{x}^i-\bar{x}^j)}}{\sum_{h=1}^{2}G_h\left[2-e^{ik_h}-e^{-ik_h}\right]+G_0}\,d\bar{k}, \\ \tilde{Z}_{ji} = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}\frac{e^{i\bar{k}\cdot(\bar{x}^j-\bar{x}^i)}}{\sum_{h=1}^{2}G_h\left[2-e^{ik_h}-e^{-ik_h}\right]+G_0}\,d\bar{k}. \end{cases} \tag{8}$$

One can observe that the first formula in Eq. (8) represents the value of the resistance between a generic node of the lattice and the node corresponding to the substrate. Furthermore, it can be noticed that the value of this resistance is invariant to any permutations of the values G_1 and G_2.

It may be interesting to calculate the resistance between the nodes $\bar{x}^i$ and $\bar{x}^j$ if the node 0 of the substrate remains disconnected. In agreement with the conventions indicated in Fig. 3, we may calculate this resistance in the following way:

$$Z_{ij} = \frac{V_i - V_j}{I_i} = \frac{\tilde{Z}_{ii}I_i + \tilde{Z}_{ij}I_j - (\tilde{Z}_{ji}I_i + \tilde{Z}_{jj}I_j)}{I_i} = \frac{\tilde{Z}_{ii}I_i - \tilde{Z}_{ij}I_i - \tilde{Z}_{ji}I_i + \tilde{Z}_{jj}I_i}{I_i} = 2\tilde{Z}_{ii} - \tilde{Z}_{ij} - \tilde{Z}_{ji}. \tag{9}$$

By considering the relations given in Eq. (8), the node–node resistance (between $\bar{x}^i$ and $\bar{x}^j$) can be written as follows:

$$Z_{ij} = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}\frac{2 - e^{i\bar{k}\cdot(\bar{x}^i-\bar{x}^j)} - e^{-i\bar{k}\cdot(\bar{x}^i-\bar{x}^j)}}{\sum_{h=1}^{2}G_h\left[2 - e^{ik_h} - e^{-ik_h}\right] + G_0}\,d\bar{k}. \tag{10}$$

For various applications that will be explained in the following sections we suppose that the nodes $\bar{x}^i$ and $\bar{x}^j$ are two adjacent nodes along the direction s of the 2D lattice ($s = 1, 2$). In this hypothesis we take into account, as main parameters of the lattice network, the node–substrate resistance and the node–node resistance between to adjacent nodes in direction s (see Fig. 1):

$$\begin{cases} R_{ns}(G_0, G_1, G_2) = \frac{1}{\pi^2}\int_0^{+\pi}\int_0^{+\pi}\frac{dk_1\,dk_2}{2G_1[1-\cos k_1]+2G_2[1-\cos k_2]+G_0}, \\ R_{nn}^s(G_0, G_1, G_2) = \frac{1}{\pi^2}\int_0^{+\pi}\int_0^{+\pi}\frac{2[1-\cos\ k_s]dk_1\,dk_2}{2G_1[1-\cos\ k_1]+2G_2[1-\cos\ k_2]+G_0}, \quad s = 1, 2. \end{cases} \tag{11}$$

3. Evaluation of the resistive lattice functions in terms of elliptic integrals

We start with the evaluation of the first integral R_{ns} in Eq. (11) performing firstly the integration over the variable k_1. This integral can be evaluated with the elementary formula (see Ref. [15]):

$$\int_0^{+\pi}\frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a-b)(a+b)}}, \quad a > |b|. \tag{12}$$

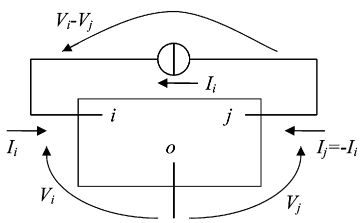

Fig. 3. Particular configuration of the network introduced to define the node–node resistance. Here the substrate node is an unconnected terminal.

Therefore, we obtain the following preliminary result:

$$R_{ns}(G_0, G_1, G_2) = \frac{1}{\pi}\int_0^{+\pi} \frac{\mathrm{d}k_2}{\sqrt{[2G_2(1-\cos k_2)+G_0][2G_2(1-\cos k_2)+G_0+4G_1]}} \tag{13}$$

Now, the remaining integration may be approached by means of the following substitution, which defines a new variable $y = 1-\cos k_2 \Rightarrow \mathrm{d}k_2 = \mathrm{d}y/\sqrt{y(2-y)}$. This operation leads to the following expression:

$$R_{ns}(G_0, G_1, G_2) = \frac{1}{2\pi G_2}\int_0^{2} \frac{\mathrm{d}y}{\sqrt{2-y}\sqrt{y}\sqrt{y+\frac{G_0}{2G_2}}\sqrt{y+\frac{G_0+4G_1}{2G_2}}}. \tag{14}$$

The last integral is of the form $\int R(\sqrt{P(x)})\,\mathrm{d}x$, where $P(x)$ is a fourth-degree polynomial; therefore, it can be reduced to elliptic integrals (see Appendix A), as follows (see Ref. [15, p. 242]):

$$\int_u^a \frac{\mathrm{d}x}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\mu, r),$$
$$a > u \geqslant b > c > d, \quad \mu = \operatorname{arcsen}\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}. \tag{15}$$

In order to apply the general solution given in Eq. (15) we define the following parameters drawing a comparison with Eq. (14): $a=2,\quad u=b=0,\quad c=-G_0/(2G_2),\quad d=-(G_0+4G_1)/(2G_2)$.Consequently, we obtain the following values for the auxiliary parameters μ and r: $\mu=\pi/2, r = 4\sqrt{G_1G_2}/\sqrt{(4G_1+G_0)(4G_2+G_0)}$. Summing up, using the relation $K(k)=F(\pi/2,\ k)$, we obtain the final result for the node–substrate resistance in 2D anisotropic networks with substrate:

$$R_{ns}(G_0, G_1, G_2) = \frac{2}{\pi\sqrt{(G_0+4G_1)(G_0+4G_2)}}\,K\left(4\sqrt{\frac{G_1G_2}{(G_0+4G_1)(G_0+4G_2)}}\right). \tag{16}$$

From now on, we take into consideration the second integral describing R_{nn}; we refer to the node–node resistance along the first spatial direction and we will obtain the other one by means of cyclic permutation of the indices. As before, a first integration in the second integral of Eq. (11) can be performed by using the property given in Eq. (12), obtaining:

$$R^1_{nn}(G_0, G_1, G_2) = \frac{1}{\pi}\int_0^{+\pi} \frac{2(1-\cos k_1)\,\mathrm{d}k_1}{\sqrt{[2G_1(1-\cos k_1)+G_0][2G_1(1-\cos k_1)+G_0+4G_2]}}. \tag{17}$$

At this point we can proceed by evaluating the integration over k_1 by means of the following substitution: $y=1-\cos k_1 \Rightarrow \mathrm{d}k_1 = \mathrm{d}y/\sqrt{y(2-y)}$. The integral is transformed in the following one:

$$R^1_{nn} = \frac{1}{\pi G_1}\int_0^{2} \frac{\sqrt{y}\,\mathrm{d}y}{\sqrt{2-y}\sqrt{y+\frac{G_0}{2G_1}}\sqrt{y+\frac{G_0+4G_2}{2G_1}}}. \tag{18}$$

Once again, this is an integration that can be reduced to elliptic integrals; we may use the following general rule (see [15, p. 265]):

$$\int_b^u \sqrt{\frac{(x-b)}{(a-x)(x-c)(x-d)}}\,\mathrm{d}x = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r)\right],$$
$$a \geqslant u > b > c > d, \quad \lambda = \operatorname{arcsen}\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}. \tag{19}$$

In order to apply Eq. (19) we define the following parameters (see terms in Eq. (18) for comparison): $a=u=2,\quad b=0,\quad c=-G_0/(2G_1),\quad d=-(G_0+4G_2)/(2G_1)$. The corresponding values for λ and r are given by $\lambda=\pi/2,\quad r=4\sqrt{G_1G_2}/\sqrt{(4G_1+G_0)(4G_2+G_0)}$. Thus a first result is given by the following expression:

$$R^1_{nn}(G_0,G_1,G_2)=\frac{2G_0}{\pi G_1\sqrt{(G_0+4G_1)(G_0+4G_2)}}\left[\Pi\left(\frac{\pi}{2},\frac{4G_1}{4G_1+G_0},r\right)-K(r)\right]. \tag{20}$$

In this result a complete elliptic integral of the third kind appears; it can be simplified by means of the following relationship that holds on when $r^2<n<1$ (this condition is known as circular case, see Ref. [16, p.599]):

$$\Pi\left(\frac{\pi}{2},n,r\right)-K(r)=\frac{1}{2}\pi\sqrt{\frac{n}{(1-n)(n-r^2)}}\left\{1-\frac{2}{\pi}\left[K(r)E(\varepsilon,\sqrt{1-r^2})-[K(r)-E(r)]F(\varepsilon,\sqrt{1-r^2})\right]\right\} \tag{21}$$

where $\varepsilon=\mathrm{arcsen}(\sqrt{1-n}/\sqrt{1-r^2})=\mathrm{arctg}(\sqrt{1-n}/\sqrt{n-r^2})$ is the argument of the incomplete elliptic integrals E and F. In our case we apply Eq. (21) with $n=4G_1/(4G_1+G_0)$. Summing up, we arrive at the following final results:

$$\begin{cases} R_{ns}(G_0,G_1,G_2)=\frac{r}{2\pi\sqrt{G_1G_2}}K(r), \\ R^1_{nn}(G_0,G_1,G_2)=\frac{1}{G_1}\left\{1-\frac{2}{\pi}[K(r)E(\varepsilon,\sqrt{1-r^2})-[K(r)-E(r)]F(\varepsilon,\sqrt{1-r^2})]\right\}, \\ R^2_{nn}(G_0,G_1,G_2)=\frac{1}{G_2}\left\{1-\frac{2}{\pi}[K(r)E(\eta,\sqrt{1-r^2})-[K(r)-E(r)]F(\eta,\sqrt{1-r^2})]\right\}, \end{cases}$$
$$\varepsilon=arctg\sqrt{\frac{4G_2+G_0}{4G_1}},\quad \eta=arctg\sqrt{\frac{4G_1+G_0}{4G_2}},\quad r=4\sqrt{\frac{G_1G_2}{(4G_1+G_0)(4G_2+G_0)}}. \tag{22}$$

The first expression in Eq. (22) is derived from Eq. (16), the second one follows from Eqs. (20) and (21) and the third formula is obtained from the second one by permuting G_1 with G_2.

Finally, we analyse the particular case of Eq. (22) concerning isotropic networks. The general expressions given in Eq. (22) may be strongly simplified when we are dealing with an isotropic network characterised by $G_1=G_2=G$. Under this hypothesis, we observe the following simplifications of the involved quantities:

$$G_1=G_2=G\Rightarrow\varepsilon=\eta=arctg\sqrt{\frac{4G+G_0}{4G}},\quad r=\frac{4G}{4G+G_0}\Rightarrow\varepsilon=\eta=arctg\sqrt{\frac{1}{r}}. \tag{23}$$

When the argument and the modulus of incomplete elliptic integrals are related as indicated in Eq. (23) some useful expressions [16] help us to handle the problem:

$$2F\left(\mathrm{arctg}\sqrt{\frac{1}{r}},\sqrt{1-r^2}\right)=K(\sqrt{1-r^2}),\quad 2E\left(\mathrm{arctg}\sqrt{\frac{1}{r}},\sqrt{1-r^2}\right)=E(\sqrt{1-r^2})+1-r. \tag{24}$$

In order to simplify the notations we define the complementary modulus as follows: $r'=\sqrt{1-r^2}$. Eq. (24) allow us to simplify the expression of the node–node resistance:

$$\begin{aligned} R_{nn}=R^1_{nn}=R^2_{nn}&=\frac{1}{G}\left\{1-\frac{2}{\pi}\left[K(r)E\left(\mathrm{arctg}\sqrt{\frac{1}{r}},r'\right)-[K(r)-E(r)]F\left(\mathrm{arctg}\sqrt{\frac{1}{r}},r'\right)\right]\right\} \\ &=\frac{1}{G}\left\{1-\frac{2}{\pi}\left[K(r)\frac{E(r')+1-r}{2}-[K(r)-E(r)]\frac{K(r')}{2}\right]\right\}=\frac{1}{2G}\left\{1-\frac{2}{\pi}(1-r)K(r)\right\}, \end{aligned} \tag{25}$$

where the Legendre relation $E(r)K(r')+K(r)E(r')-K(r)K(r')=\pi/2$ for complete elliptic integrals has been used. So, final formulas for the node–substrate and node–node resistance for isotropic networks follow:

$$\begin{cases} R_{ns}(G_0,G,G)=\frac{2}{\pi(4G+G_0)}K\left(\frac{4G}{4G+G_0}\right) \\ R_{nn}(G_0,G,G)=\frac{1}{2G}\left\{1-\frac{2G_0}{\pi(4G+G_0)}K\left(\frac{4G}{4G+G_0}\right)\right\} \end{cases} \tag{26}$$

Expression for R_{ns} immediately follows from Eq. (22) when the value of the modulus r is taken from Eq. (23). The relationship for R_{nn} is the explicit version of that obtained in Eq. (25).

It is interesting to note that, in isotropic networks, R_{ns} and R_{nn} are related by the simple relationship $R_{nn}=(2G)^{-1}(1-G_0R_{ns})$.

4. Theory for random networks

We shall refer ourselves to 2D lattice networks with substrate and we define an anisotropic distribution of conductance values by introducing different probability densities $\rho_k(G)$ for the conductances aligned along the different lattice directions k ($k=1, 2$). Moreover, the conductances of the substrate are distributed following a given probability density $\rho_0(G)$. All the conductance values (each direction and substrate) are independently distributed according to the probability densities upon described. In the effective medium theory the average effects of the random conductances, in such a disordered network, will be represented by an anisotropic effective network in which all the conductances in k direction ($k=1, 2$) have the same value $\bar{G}_k$ and all the conductances in the substrate have the same value $\bar{G}_0$. These effective conductances will be self-consistently determined by the requirement that the fluctuating local potential in the random network should average to zero. This is the main idea, which allows us to build up the effective medium theory. Furthermore, for following purposes, we suppose to be able to evaluate the functions $R_{nn}^k=R_{nn}^k(\bar{G}_0,\bar{G}_1,\bar{G}_2)\forall k=1,2$ and $R_{ns}=R_{ns}(\bar{G}_0,\bar{G}_1,\bar{G}_2)$ (Eq. (22) or Eq. (26) for isotropic networks) that play a crucial role in determining the effective network equivalent to a random one. From now on, we suppose to know the effective network corresponding to a given random one, in order to understand the conceptual connection among them. In the effective network we change a single conductance $\bar{G}_k$, oriented along the direction k ($k=1, 2$) or belonging to the substrate ($k=0$), back to its true value G_k. This procedure can be applied indifferently to a resistor in the lattice or a resistor in the substrate. In Fig. 4 one can find the graphical representation of such a substitution, where the Thevenin equivalent circuit of the remaining part of the effective network is indicated. Here, G_k is a particular instance for the conductance value and $\bar{G}_k$ is the corresponding effective value. Moreover, V_{eq} and G_{eq} are the parameters of the Thevenin equivalent circuit. The electrical potentials in the circuits described in Fig. 4 can be evaluated as follows:

$$\bar{V}=V_{eq}\frac{G_{eq}}{G_{eq}+\bar{G}_k},\quad V=V_{eq}\frac{G_{eq}}{G_{eq}+G_k}. \tag{27}$$

So, the potential fluctuations due to the random character of the network are given by

$$\Delta V=V-\bar{V}=V_{eq}\frac{G_{eq}}{G_{eq}+G_k}-\bar{V}=\bar{V}\frac{G_{eq}+\bar{G}_k}{G_{eq}}\frac{G_{eq}}{G_{eq}+G_k}-\bar{V}=\bar{V}\frac{\bar{G}_k-G_k}{G_{eq}+G_k}. \tag{28}$$

Now, we may observe a relationship between the Thevenin conductance G_{eq} and the values R_{nn} and R_{ns}. In fact, G_{eq} is the conductance between the nodes **A** and **B** of Fig. 4, where we have eliminated the conductance $\bar{G}_k$. Thus, in agreement with Fig. 5, we may write down the relations:

$$\begin{cases} k=1,2\Rightarrow R_{nn}^k=\frac{1}{G_{eq}+\bar{G}_k}\Rightarrow G_{eq}=\frac{1}{R_{nn}^k}-\bar{G}_k, \\ k=0\Rightarrow R_{ns}=\frac{1}{G_{eq}+\bar{G}_0}\Rightarrow G_{eq}=\frac{1}{R_{ns}}-\bar{G}_0. \end{cases} \tag{29}$$

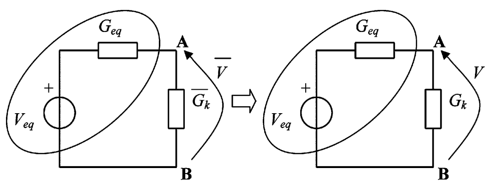

Fig. 4. Thevenin equivalent circuit between two adjacent arbitrary nodes of the lattice before and after the substitution of the effective conductance $\bar{G}_k$ with a particular random instance G_k.

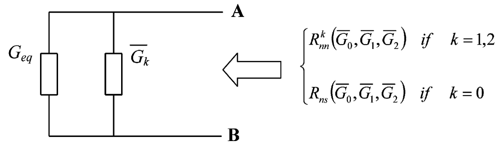

Fig. 5. Scheme that defines the relation between the Thevenin conductance G_{eq} introduced in Fig. 4 and the characteristic resistance values $R^k_{nn} = R^k_{nn}(\bar{G}_0, \bar{G}_1, \bar{G}_2)\forall k$ and $R_{ns} = R_{ns}(\bar{G}_0, \bar{G}_1, \bar{G}_2)$.

By substituting Eq. (29) into Eq. (28) we immediately obtain:

$$k = 1,2 \Rightarrow \Delta V = \bar{V}\frac{\bar{G}_k - G_k}{1/R^k_{nn} - \bar{G}_k + G_k}, \quad k = 0 \Rightarrow \Delta V = \bar{V}\frac{\bar{G}_0 - G_0}{1/R_{ns} - \bar{G}_0 + G_k}. \tag{30}$$

By imposing that the fluctuations of the potential average to zero, we have the homogenising integral equations:

$$\begin{cases} \int_0^{+\infty} \rho_k(G)\frac{\bar{G}_k - G}{1/R^k_{nn}(\bar{G}_0,\bar{G}_1,\bar{G}_2) - \bar{G}_k + G}\,\mathrm{d}G = 0\forall k = 1,2, \\ \int_0^{+\infty} \rho_0(G)\frac{\bar{G}_0 - G}{1/R_{ns}(\bar{G}_0,\bar{G}_1,\bar{G}_2) - \bar{G}_0 + G}\,\mathrm{d}G = 0. \end{cases} \tag{31}$$

These relations represent a system of three equations with three unknowns $\bar{G}_0, \bar{G}_1, \bar{G}_2$ that can be found when all the probability densities involved are given. The expressions for the resistance R_{nn} and R_{ns} in terms of the effective conductances $\bar{G}_0, \bar{G}_1, \bar{G}_2$ are given in Eq. (22) (or Eqs. (26) for isotropic networks). The analysis of some random systems with the help of Eq. (31) will be described in the following sections, drawing a comparison between theoretical results and Monte-Carlo simulations.

Some comments follow about the role of the previous results in the context of the literature on the characterization of random networks. The model developed is a self-consistent effective medium theory based on a mean field procedure and therefore it has the advantage to furnish explicit homogenising schemes for several situations where the effects of a substrate on a 2D random system must be taken into account. A different analysis could be performed when one is interested in obtaining the effective conductances as a power series in the contrast between the constituents. In such a case the results can be obtained by means of the introduction of graphs, which allow the calculation of all the contribution of a given order [10]. One can verify (by repeating the calculation performed in Ref. [10] for 2D system) that our results are in perfect agreement with this approach at least up to the third order.

5. Simulations for resistive networks

We devote our attention to some paradigmatic cases of 2D networks with substrate that are interesting for practical applications. The first case deals with a 2D isotropic grid where the conductances of the lattices are randomly placed and the conductances towards the substrate are all fixed at a given value. Each conductance of the grid is placed into the network assuming the value G_1 with probability 1/2 and the value G_2 with probability 1/2. All the substrate conductances are fixed to the value G_0. The combination of Eqs. (31) and (26) leads, after some simple manipulations, to the following equation for the unknown effective conductance $\bar{G}$ of the grid:

$$K\left(\frac{4\bar{G}}{4\bar{G} + G_0}\right) = \frac{\pi}{2}\frac{(G_1G_2 - \bar{G}^2)(4\bar{G} + G_0)}{G_0(\bar{G} - G_1)(\bar{G} - G_2)}. \tag{32}$$

Of course, in Eq. (31) we have taken into account only the first relation describing the effective conductances in the grid. This result is interesting because the parameter G_0 modulates the kind of mean value $\bar{G}$ between G_1 and G_2. The extreme cases are the following. If G_0 is zero the value $\bar{G}$ corresponds to the *geometric mean* between G_1 and G_2, $\bar{G} = \sqrt{G_1G_2}$, and if G_0 tends to infinity the value $\bar{G}$ corresponds to the *arithmetic mean* between G_1 and G_2, $\bar{G} = (G_1 + G_2)/2$. In other words, Eq. (32) defines a family of mean values that depend on the value of G_0. Simulations with $G_1 = 1$ and $G_2 = 4$ have been performed. A comparison between results obtained from Eq. (32) and Monte-Carlo simulations is shown in Fig. 6 where theoretical values of R_{ns} and numerical ones are shown versus values of G_0 between 1 and 100. A remarkably good fitting is evident.

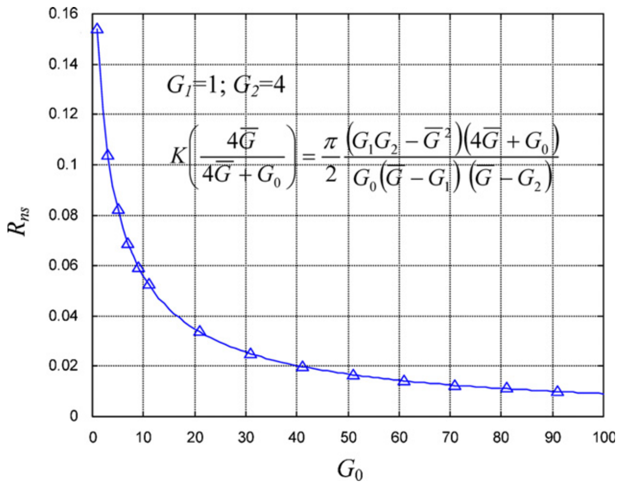

Fig. 6. Theoretical (continuous line) and Monte-Carlo results (triangles) for a grid where the conductances of the lattices are randomly placed and the conductances of the substrate are all fixed at a given value G_0. Each conductance of the grid is placed into the network assuming the value 1 with probability 1/2 and the value 4 with probability 1/2. The node–substrate resistance R_{ns} of the network is represented versus the values of G_0.

Furthermore, it may be interesting to note that, by using properties (A.5) and (A.6) of the appendix, Eq. (32) may be written in terms of the *arithmetic–geometric mean* $M(a, b)$ as follows:

$$M\left(1, 1+8\frac{\bar{G}}{G_0}\right)=\frac{(\bar{G}-G_1)(\bar{G}-G_2)}{G_1G_2-\bar{G}^2}. \tag{33}$$

This equation reveals a strong conceptual connection between averaging processes in random networks and the *arithmetic–geometric mean* procedure.

The second case of 2D resistive lattices deals with a grid with fixed conductances in both directions and random conductances in the substrate. Each conductance of the substrate is placed into the network assuming the value G_1 with probability 1/2 and the value G_2 with probability 1/2. All the lattice conductances are fixed to the value G (in both directions). As before, the combination of Eqs. (31) and (26) leads, after some simple manipulations, to the following equation for the unknown effective conductance $\bar{G}_0$ of the grid:

$$K\left(\frac{4G}{4G+\bar{G}_0}\right)=\frac{\pi}{2}\frac{\left(\bar{G}_0-\frac{G_1+G_2}{2}\right)(4G+\bar{G}_0)}{(\bar{G}_0-G_1)(\bar{G}_0-G_2)}. \tag{34}$$

In Eq. (31) we have taken into account only the second relation describing the effective conductances in the substrate. This result is similar to that given in Eq. (32) because the parameter G modulates the kind of mean value $\bar{G}_0$ between G_1 and G_2 also in this case; the limiting cases are the following: if G is zero the value $\bar{G}_0$ corresponds to the *harmonic mean* between G_1 and G_2, $\bar{G}_0 = 2G_1G_2/(G_1+G_2)$, and if G tends to infinity the value $\bar{G}_0$ corresponds to the *arithmetic mean* between G_1 and G_2, $\bar{G}_0 = (G_1+G_2)/2$. Finally, simulations with $G_1 = 1$ and $G_2 = 4$ have been performed. A comparison between results obtained from Eq. (34) and Monte-Carlo simulations is shown in Fig. 7 where theoretical values of R_{ns} and numerical ones are shown versus values of G between 1 and 100. Once again, the theory is in very good agreement with numerical results. As before, we note that Eq. (34) may be written in terms of the *arithmetic–geometric mean* as follows:

$$M\left(1, 1+8\frac{G}{\bar{G}_0}\right)=\frac{(\bar{G}_0-G_1)(\bar{G}_0-G_2)}{\bar{G}_0\left(\bar{G}_0-\frac{G_1+G_2}{2}\right)}. \tag{35}$$

These cases can be considered as paradigmatic random systems that have immediate applications to the analysis of heterogeneous films deposited on substrates with an electrical coupling. Such cases have been

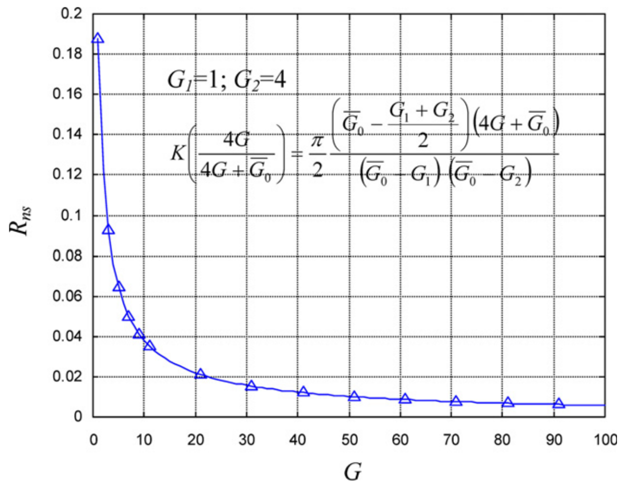

Fig. 7. Theoretical (continuous line) and Monte-Carlo results (triangles) for a grid where the conductances G of the lattices are fixed and the conductances of the substrate are randomly assigned. Each conductance toward the substrate is placed into the network assuming the value 1 with probability 1/2 and the value 4 with probability 1/2. All the grid conductances are fixed to the value G. The node–substrate resistance R_{ns} of the network is represented versus the values of G.

analysed under the hypothesis of ohmic coupling between the deposited film and the substrate. Some generalisations will be described in the following section.

6. Capacitive substrate

For practical applications it could be interesting to consider a capacitive coupling between the random grid and the random substrate. This aspect is important to model heterogeneous films randomly deposited on substrate when capacitive effects between the film and the substrate medium appear and are not negligible. Here, we devote our attention on a generalisation of the previous results, which may be applied to the present case of capacitive substrate. Firstly, we consider a random resistor grid with a constant capacitive coupling with the substrate. We suppose that each conductance of the lattice is placed into the network assuming the value G_1 with probability 1/2 and the value G_2 with probability 1/2. All the vertical components are assumed to be capacitors of fixed value C. Therefore each vertical component has a frequency-dependant admittance equals to $i\omega C$. We suppose that Eq. (32) continues to be correct when conductances are substituted with complex valued admittances. So, we search for the equivalent admittance $\bar{Y}$ of the grid with the following equation:

$$K\left(\frac{4\bar{Y}}{4\bar{Y}+i\omega C}\right)=\frac{\pi}{2}\frac{(G_1G_2-\bar{Y}^2)(4\bar{Y}+i\omega C)}{i\omega C(\bar{Y}-G_1)(\bar{Y}-G_2)}. \tag{36}$$

This relation is more complicated than the counterpart, given in Eq. (32), because of the presence of the elliptic integral calculated with a complex modulus: its numerical computation has been carried out by using the complex arithmetic–geometric mean procedure as described in the appendix. In fact, it is known that the convergence of the arithmetic–geometric procedure is assured also with complex variables [17]. Moreover, we observe that the resulting $\bar{Y}$ is a complex valued function of the frequency. When Eq. (36) is solved for the equivalent admittance $\bar{Y}$ we may calculate the average node–substrate impedance by means of the following (see Eq. (26)):

$$Z_{ns}=\frac{2}{\pi(4\bar{Y}+i\omega C)}K\left(\frac{4\bar{Y}}{4\bar{Y}+i\omega C}\right). \tag{37}$$

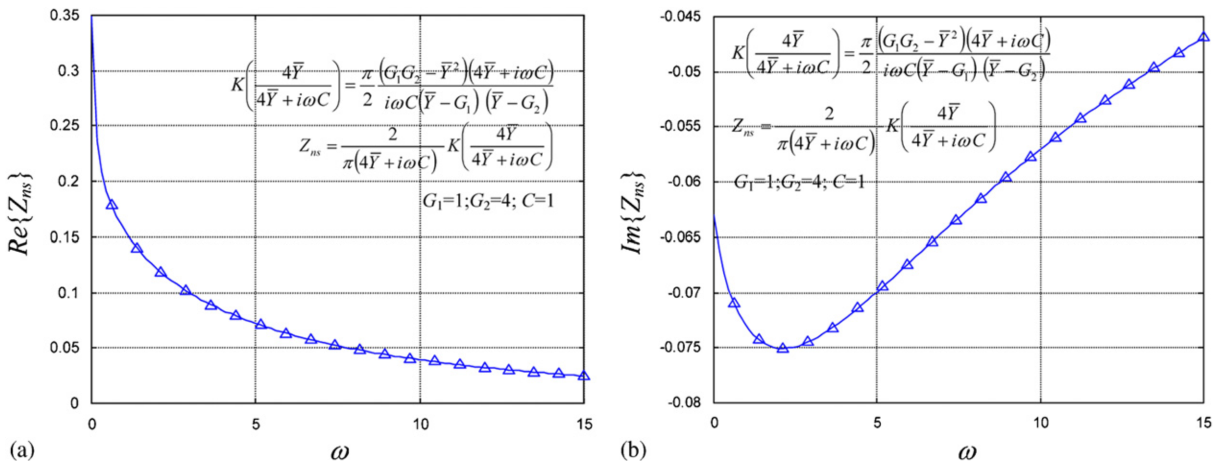

Fig. 8. Real (a) and imaginary (b) part of the node–substrate impedance Z_{ns} versus the values of ω: comparison between theoretical (continuous line) and Monte-Carlo results (triangles). The plots refer to a grid where the conductances of the lattices are randomly placed and the capacitors towards the substrate are all fixed at a given value $C = 1$. Each conductance of the grid is placed into the network assuming the value 1 with probability 1/2 and the value 4 with probability 1/2.

In Fig. 8 one can find a comparison between the real and imaginary parts of the node–substrate impedance computed with the above stated theoretical considerations and with Monte-Carlo simulations; a remarkably good agreement has been found out. In this simulation we have used the values $G_1 = 1$, $G_2 = 4$, $C = 1$ and we have studied the system in the frequency range $0<\omega<15$.

Now, we consider a second example, dual of the previous one, where all the conductances of the grid are fixed to the same value G and the capacitive coupling with the substrate is randomly generated. In particular, all the vertical components have been assumed as capacitors with value C_1 with probability 1/2 and value C_2 with probability 1/2. The equation that allows us to compute the equivalent vertical admittance $\bar{Y}_0$ is a generalisation of Eq. (34):

$$K\left(\frac{4G}{4G+\bar{Y}_0}\right) = \frac{\pi}{2}\,\frac{\left(\bar{Y}_0 - \mathrm{i}\omega\,\frac{C_1+C_2}{2}\right)(4G+\bar{Y}_0)}{(\bar{Y}_0-\mathrm{i}\omega C_1)(\bar{Y}_0-\mathrm{i}\omega C_2)}. \tag{38}$$

Once again, $\bar{Y}_0$ is a complex valued function of the frequency that may be obtained by correctly compute elliptic integral with complex modulus. When a value for $\bar{Y}_0$ is found the corresponding node–substrate impedance is given by (see Eq. (26)):

$$Z_{ns} = \frac{2}{\pi(4G+\bar{Y}_0)}\,K\left(\frac{4G}{4G+\bar{Y}_0}\right). \tag{39}$$

Numerical experiments have been performed for such kind of networks: in Fig. 9 one can find a comparison between the real and imaginary parts of the node–substrate impedance computed with Eqs. (38) and (39) and with Monte-Carlo simulations; once again, a remarkably good agreement has been obtained. In such simulation we have used the values $C_1 = 1$, $C_2 = 4$, $G = 1$ and we have studied the system in the frequency range $0<\omega<16$.

Finally, a further random structure of the network has been taken into consideration: each conductance of the lattice is placed in the system assuming the value 0 with probability p and value G with probability $1-p$. This percolative structure has been connected to the substrate with a capacitive constant coupling: it means that each vertical component is a capacitor with fixed value C (admittance $\mathrm{i}\omega C$). Thus, the network should exhibit percolation (with percolation threshold at $p_c = 1/2$) for very low values of the frequency and this effect should disappear for high values of the frequency. In fact, in very low frequency condition, the admittances towards the substrate are negligible and the substrate is not influent. In other words we are studying a

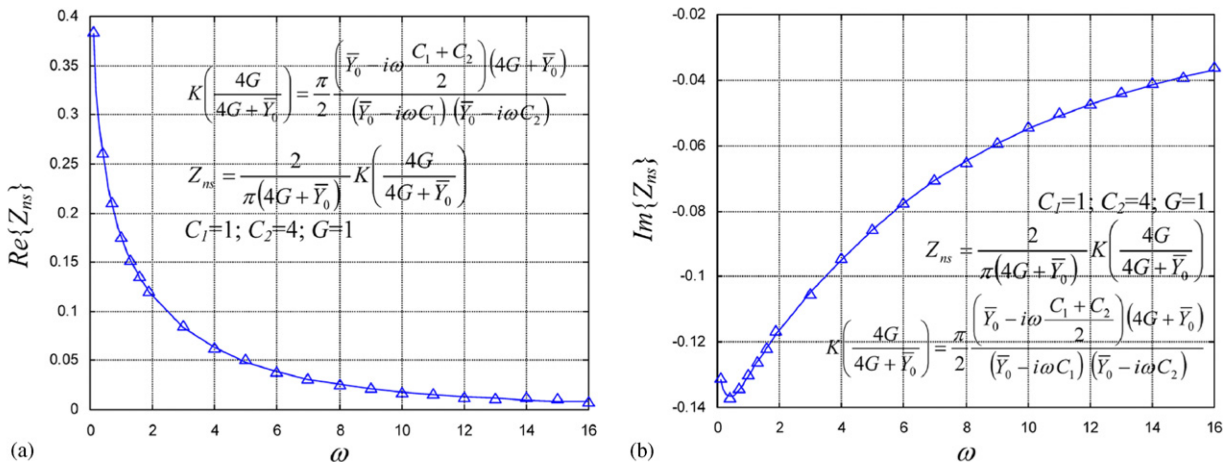

Fig. 9. Real (a) and imaginary (b) part of the node–substrate impedance Z_{ns} versus the values of ω: comparison between theoretical (continuous line) and Monte-Carlo results (triangles). The plots refer to a grid where the conductances of the lattices are all fixed at a given value $G = 1$ and the capacitors towards the substrate are randomly placed. Each capacitor is placed into the network assuming the value 1 with probability 1/2 and the value 4 with probability 1/2.

percolative system controlled by the frequency. The theoretical description of the behaviour of this system is based on Eqs. (31) and (26), which lead, after some straightforward calculations, to the following homogenising result:

$$K\left(\frac{4\bar{Y}}{4\bar{Y}+\mathrm{i}\omega C}\right)=\frac{\pi}{2}\frac{[(1-2p)G-\bar{Y}](4\bar{Y}+\mathrm{i}\omega C)}{\mathrm{i}\omega C(\bar{Y}-G)}. \tag{40}$$

Here, $\bar{Y}$ is the complex valued, frequency dependant, equivalent admittance for the lattice network. When $\omega \to 0$ it is easy to observe that the solution for $\bar{Y}$ is real and is given by: $\bar{Y}=(1-2p)G$ if $0<p<1/2$ and $\bar{Y}=0$ if $1/2<p<1$. This is the classical percolation behaviour which is exhibited for a system without substrate, where the percolation threshold assume the value $p_c = 1/2$. When $\omega > 0$ the effective admittance $\bar{Y}$ become a complex valued function which takes into account the effects of the capacitive substrate. As before, the knowledge of $\bar{Y}$ allows us to evaluate the average node–substrate impedance, as follows:

$$Z_{ns}=\frac{2}{\pi(4\bar{Y}+\mathrm{i}\omega C)}K\left(\frac{4\bar{Y}}{4\bar{Y}+\mathrm{i}\omega C}\right). \tag{41}$$

The analytical description given by Eqs. (40) and (41) has been tested with Monte-Carlo simulations obtaining the results shown in Figs. 10 and 11. We have considered the values $G = 1$, $C = 1$ and we have studied the system's behaviour versus the stoichiometric parameter p and the frequency ω. In Fig. 10 real and imaginary parts of $\bar{Y}$ in logarithmic scale (solution of Eq. (40)) are shown in terms of p and for different values of the frequency: the comparison between theory and simulations is very good. Finally, in Fig. 11 real and imaginary parts of Z_{ns} (Eq. (41)) are shown in terms of p and for different values of ω: once again, the comparison between theory and simulations is remarkably good. We may observe that the imaginary part of the effective admittance $\bar{Y}$ assume its maximum value, at a given frequency, in correspondence to the percolation threshold $p_c = 1/2$.

7. Conclusions

We have developed an effective medium theory for a general disordered lattice network of resistors. The theoretical predictions have been verified by means of a series of Monte-Carlo analyses with remarkably good agreement. The general approach takes into account a new topology of networks that describe the effects of a substrate coupled to the resistors lattice. This is an important point in many applications such as films of

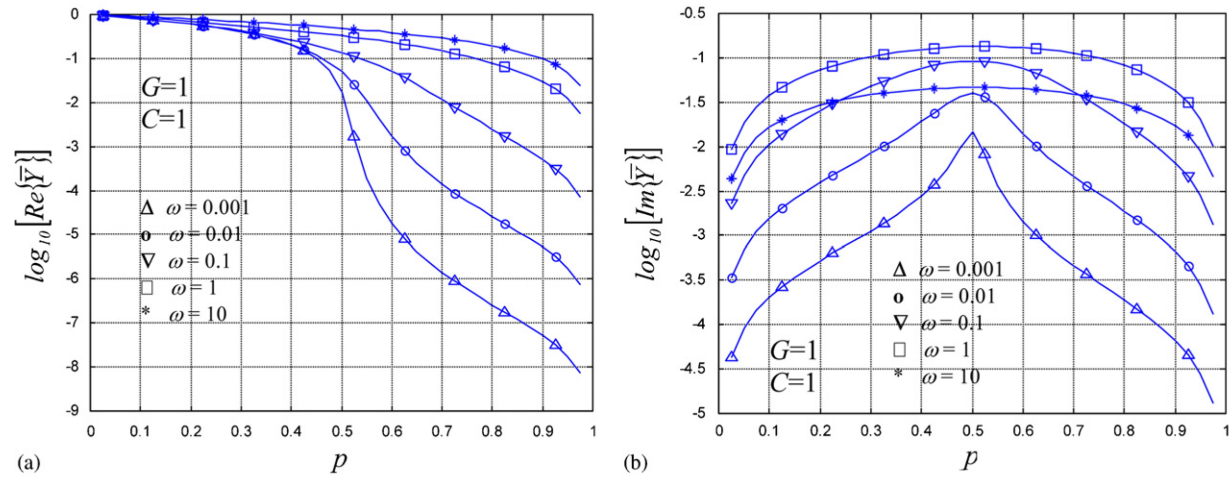

Fig. 10. Real (a) and imaginary (b) parts of the equivalent admittance $\bar{Y}$ (in logarithmic scale) for the frequency controlled percolative network. The graphs are shown in terms of the probability p for different values of the frequency. A comparison between theoretical (continuous line) and Monte-Carlo results (symbols) has been drawn. In the case of extremely low frequency the breakdown of the real part appearing for $p = 1/2$ represents the percolation phenomenon.

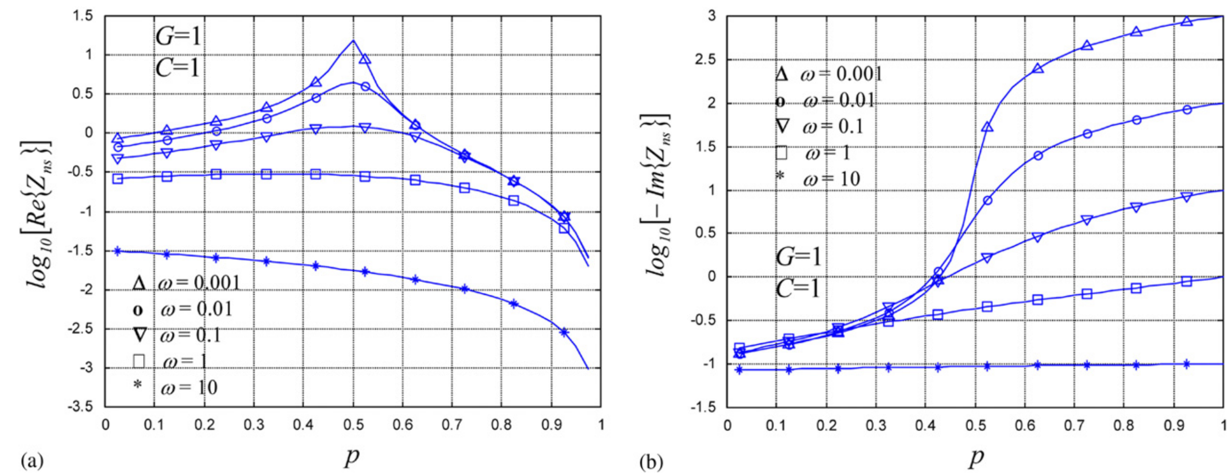

Fig. 11. Real (a) and imaginary (b) parts of the node substrate impedance Z_{ns} (in logarithmic scale) for the frequency controlled percolative network. The graphs are shown in terms of the probability p for different values of the frequency. A comparison between theoretical (continuous line) and Monte-Carlo results (symbols) has been drawn.

heterogeneous material deposited on substrates with electrical interaction. An interesting field of application of the present theory concerns with the effects of the capacitive coupling with the substrates. Several examples of lattices have been described in the text. Moreover, as additional results we have reported many exact relations describing the electrical behaviour of homogeneous but anisotropic infinite lattice systems. These results are based on the Green's lattice function expressed in terms of elliptic integrals and may be useful for different applications.

Appendix. Definition of the complete and incomplete elliptic integrals

Every integral of the form $\int R(x, \sqrt{P(x)})\,\mathrm{d}x$, where $P(x)$ is a third- or fourth-degree polynomial can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals (see Refs. [15,16]):

$$\begin{cases} F(\varphi,k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\,sen^2\,\alpha}} = \int_0^{sen\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\ E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\,sen^2\,\alpha}\,d\alpha = \int_0^{sen\varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx \\ \Pi(\varphi,n,k) = \int_0^{\varphi} \frac{d\alpha}{(1-n\,sen^2\,\alpha)\sqrt{1-k^2\,sen^2\,\alpha}} = \int_0^{sen\varphi} \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}} \end{cases} \tag{A.1}$$

which are called, respectively, incomplete elliptic integrals of the first, second and third kind in the trigonometric form and in the Legendre form. The number k is called the modulus of these integrals, the number $\sqrt{1-k^2}=k'$ is called the complementary modulus and the number n is the parameter of the integral of the third kind. Elliptic integrals from 0 to $\pi/2$ are called complete elliptic integrals:

$$K(k) = F\left(\frac{\pi}{2},k\right), \quad E(k) = E\left(\frac{\pi}{2},k\right). \tag{A.2}$$

For the properties of the complete elliptic integral of the third kind see Ref. [16]. Moreover, in the main text, we have made use of the complete elliptic integral of the first kind $K(k)$ with complex modulus k; it is defined by

$$K(k) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1,k^2\right) = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\cdots+\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2k^{2n}+\cdots\right\}. \tag{A.3}$$

The hypergeometric function F converges absolutely in the disk $|k|<1$ and it is continued analytically. The numerical computation of $K(k)$ with complex modulus has been performed by means of the arithmetic–geometric mean procedure $M(a, b)$ with complex variables, defined by the following iterations:

$$\begin{cases} a_0 = a, \\ a_N = \frac{1}{2}(a_{N-1}+b_{N-1}), \end{cases} \begin{cases} b_0 = b, \\ b_N = \sqrt{a_{N-1}b_{N-1}}. \end{cases} \tag{A.4}$$

When N approaches infinity a_N and b_N converge to the mean $M(a, b)$. For the square root evaluation, the branch where the argument of b_N is between $-\pi/2$ and $\pi/2$, must always be chosen in the course of the calculation. The complete elliptic integral of the first kind (with complex modulus) is related to the arithmetic–geometric mean by the following [17]:

$$K(k) = \frac{\pi}{2M(1,\sqrt{1-k^2})} = \frac{\pi}{2M(1,k')}. \tag{A.5}$$

Some other properties of $M(a, b)$ may be useful (i.e. for obtaining the alternative Eqs. (33) and (35)):

$$M(a,a)=a, \quad M(a,b) = M\left(\frac{a+b}{2},\sqrt{ab}\right), \quad M(\lambda a,\lambda b) = \lambda M(a,b). \tag{A.6}$$

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