International Journal of Applied Electromagnetics and Mechanics 26 (2007) 1–19
IOS Press

Electrical behaviour of a single crack in a conductor and exponential laws for conductivity in micro cracked solids

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Abstract. In this paper we analyse the electrical effects of the presence of cracks in solid conductors. We have studied such problematic from different points of view. Firstly, we have analytically evaluated the behaviour of the electrical field near a crack in an isotropic solid where a uniform current density is flowing, drawing a comparison with the behaviour of the well known stress and strain tensor fields in the analogue elastic problem. This computation has been made with a slit-crack (two-dimensional field analysis) and with a circular crack (three-dimensional field analysis). So, in order to quantify the spatial fluctuations of the local electric field around the crack we have numerically found the density of states for the field showing that it exhibit sharp peaks and abrupt changes in the slope at certain critical points which are analogous to van Hove singularities in the density of states for phonons and electrons in solids. Finally, we have performed a theoretical analysis of the conductivity of a microcracked solid. The distribution of cracks in the solid follows a given orientational distribution, which modify the conduction properties of the overall material. In particular, we have shown that the conductivity depends exponentially on the cracks density and on the size of each crack embedded in the medium.

Keywords: Electric cracks, field intensity factors, density of states, order parameters

1. Introduction

Fracture mechanics is one of the most heavily developed branches of engineering science and applied mathematics [1–3]. There are two lines of research in order to study the behaviour of cracks in materials. The first one concerns with continuum fracture mechanics. In this case the general strategy is to solve the displacement fields in the medium subjected to both the boundary conditions and the externally applied stress. The second line of research is the attempt to understand the crack behaviour at atomic level by using molecular dynamics simulations [4]. Results concerning the stress behaviour near a crack, well described by the stress intensity factors, are in perfect agreement between these approaches [5,6]. In this work we study the electrical behaviour of a crack in a given conductor by means of the continuum theory standpoint. We will show that many results well known in the field of the mechanical behaviour of cracks hold also for the electric behaviour. For example, the stress intensity factor that describe the singular behaviour of the stress field near the crack tips can be translated in the electric case obtaining an electric field intensity factor. We considered two different shape of crack: a slit-crack represented in Fig. 1 and a circular crack represented in Fig. 2. For these two kinds of crack, the standard stress

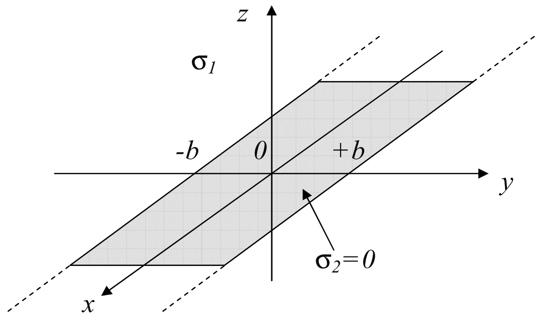

Fig. 1. Geometrical representation of a slit-crack lying on the plane $x - y$ and aligned along the x-axis. The half-length of the crack is b.

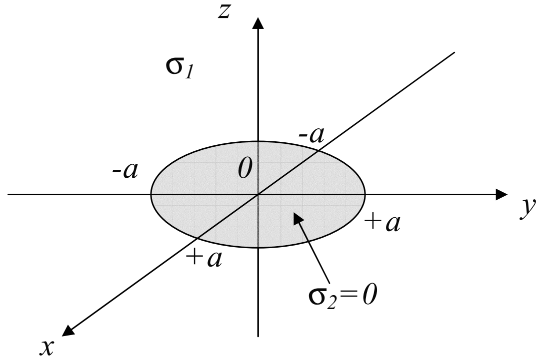

Fig. 2. Geometrical representation of a circular crack lying on the plane $x - y$. The radius of the crack is a.

intensity factors are given by the following relations [1,7,8]. With the geometry defined in Fig. 1, the distance for the tip of the slit-crack is given by $\eta = y - b$ where b is the semi aperture of the crack. When a tensional stress $\sigma_{zz,\infty}$ is applied along the z-axis to the structure (mode I), the singular behaviour of the stress field near the tip crack is described by the following stress intensity factor:

$$K_I = \lim_{\substack{\eta \to 0 \\ z \to 0}} \sqrt{2\pi\eta}\sigma_{zz}(\eta, z) = \sqrt{b\pi}\sigma_{zz,\infty} \tag{1}$$

Similarly for a circular crack (see Fig. 2) the distance from the border of the crack is given by $\eta = \sqrt{x^2 + y^2} - a$ where a is the radius of the crack. The stress intensity factor is given by:

$$K_I = \lim_{\substack{\eta \to 0 \\ z \to 0}} \sqrt{2\pi\eta}\sigma_{zz}(\eta, z) = \frac{2\sqrt{a}}{\sqrt{\pi}}\sigma_{zz,\infty} \tag{2}$$

In the first section of this work we analyse the perturbation to the electric potential due to the presence of a crack in a conducting medium in which a given uniform current density is flowing. We derived exact expressions giving the electric potential around the crack, which are valid in the entire space. In particular, we will show that the electric field at the tips is singular with a behaviour very similar to that explained by Eqs (1) or (2).

From the electric point of view a crack is a region of a plane where the electric current cannot flow. In the present work, in order to model the flat shape of a crack, we adopt an ellipsoidal void (zero conductivity) with an axis with infinitesimal length. Treating the crack as a vacuous oblate ellipsoid of eccentricity approaching zero is very convenient. The idea is that one can derive the needed formulas for a microcracked solid, passing (with a due care) to the limits in the general formulas, concerning ellipsoidal inclusions [9,10]. This approach is not new in principle and it has been used in various homogenisation theories for microcracked media from both electric and mechanic point of views [11–13]. In this work, this approach is firstly applied to analyse the electric quantities in a region where a single crack is present. Moreover, in order to study the local electric field fluctuations around a given crack we have analyzed the density of states for the electric field [14,15]. We have numerically found the presence of some singularities in the distribution function of the intensity of the electric field in the region around the crack. Van Hove in a famous work [16] showed that for a crystal, the frequency distribution function of elastic vibrations has analytic singularities. These singularities are very similar to that here obtained for the electric field and are very useful in characterizing field fluctuations in materials with defects and inclusions.

Finally, we have theoretically analysed the effects of the presence of a given distribution of cracks on the conductivity of a solid. In earlier literature many works have been devoted to the study of this topic [17,18]. In such relevant works the orientational distribution of cracks is given by one of the two most adopted distributions: cracks aligned with a given direction or cracks uniformly oriented in the space. The aim of the present study is that of analysing a microcracked solid with an arbitrary angular distribution of cracks. So, a particular attention is devoted to the analysis of the effects of the orientational distribution of the cracks inside the damaged material on the overall conductivity. The limiting cases of the present theory are represented by all the particles aligned with a given direction (order) and all the particles randomly oriented (disorder) [19]. We take into account all the intermediate configurations between order and disorder with the aim to characterise a material with cracks partially aligned. Two different cases have been taken into consideration: the two-dimensional distribution of slit-cracks and the three-dimensional distribution of circular cracks. In Fig. 3 one can find some orientational distributions between the upon-described limiting cases, in 2D electrostatics. The angular distribution of cracks is statistically well described by an order parameter P. Similarly, in Fig. 4 it has been shown different orientational distributions of circular cracks in 3D electrostatics. In such a case, another order parameter S defines the orientational distribution of cracks. The mathematical details will be discussed later on.

2. Single crack in a conductor: Electrical behaviour

In this section we analyse the behaviour of the electrical potential and electrical field around a crack in a conducting medium exposed to uniform electric field. The uniform density current induced in the medium is perturbed by the presence of the crack, which cannot conduct electric current. We analyse the electrical potential and the electric field behaviour in two case: a slit crack (see. Fig. 1) and a circular crack (see Fig. 1). The theory, in both cases, is based on the following preliminary result, which describes the behaviour of an ellipsoidal particle (σ_2) embedded in a homogeneous medium (σ_1) [20]. Let the axes of the ellipsoid be a_x, a_y and a_z (aligned with axes x, y, z of the reference frame) and let a uniform electrical field $\overline{E_0} = (E_{0x}, E_{0y}, E_{0z})$ applied to the structure. If the ellipsoid is absent a uniform current density $\bar{J}_0 = \sigma_1 \overline{E_0}$ is induced in the region. When the ellipsoid is present the current lines are modified

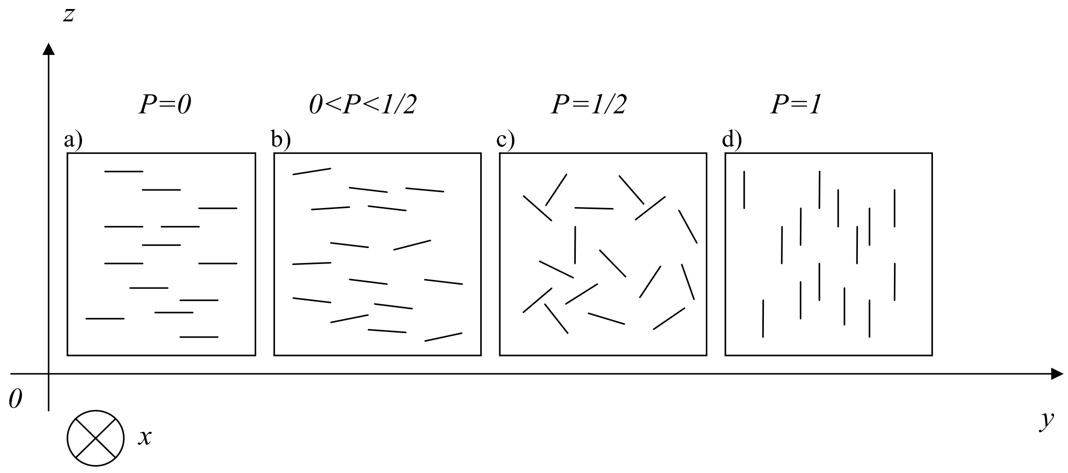

Fig. 3. Structure of a microcracked solid conductor with slit-cracks in 2D electrostatics. One can find some orientational distributions ranging from order to disorder. The order parameter P is indicated.

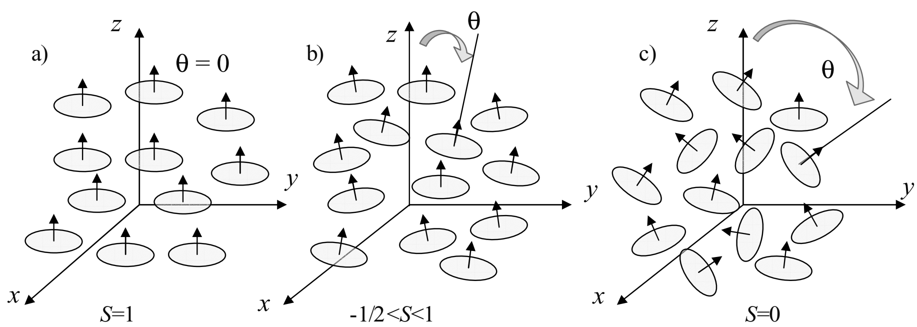

Fig. 4. Structure of a microcracked solid conductor with circular cracks in 3D electrostatics. One can find some orientational distributions ranging from order to disorder. The order parameter S is indicated.

as described in the following. However, according to Stratton [20] the electric field inside the ellipsoid is uniform and it can be computed as follows. We define the function:

$$R(s) = \sqrt{\left(s + a_x^2\right)\left(s + a_y^2\right)\left(s + a_z^2\right)} \tag{3}$$

and the depolarisation factors along each axes (k ranges over the symbols x, y and z):

$$L_k = \frac{a_x a_y a_z}{2} \int_0^{+\infty} \frac{ds}{\left(s + a_k^2\right) R(s)} \tag{4}$$

We may observe that $L_x + L_y + L_z = 1$. Therefore, the electrical field inside the ellipsoid is given, in components, by [20] (k ranges over the symbols x, y and z):

$$E_{i,k} = \frac{\sigma_1 E_{0k}}{\sigma_1 + L_k \left(\sigma_2 - \sigma_1\right)} \tag{5}$$

Of course, also the internal current density is uniform and it is given by $J_{i,k} = \sigma_2 E_{i,k}$. Moreover, the perturbation of the electric potential outside the ellipsoid can be evaluated by means of the exact result [20]:

$$\begin{aligned}\varphi(x,y,z) = &-E_{0x}x\frac{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{\xi}\frac{ds}{(s+a_x^2)R(s)}}{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{+\infty}\frac{ds}{(s+a_x^2)R(s)}} - E_{0y}y\frac{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{\xi}\frac{ds}{\left(s+a_y^2\right)R(s)}}{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{+\infty}\frac{ds}{\left(s+a_y^2\right)R(s)}} + \\ &-E_{0z}z\frac{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{\xi}\frac{ds}{(s+a_z^2)R(s)}}{1+\frac{\sigma_2-\sigma_1}{\sigma_1}\frac{a_xa_ya_z}{2}\int\limits_0^{+\infty}\frac{ds}{(s+a_z^2)R(s)}}\end{aligned} \tag{6}$$

where the variable ξ is defined by means of the following family of confocal ellipsoids:

$$\frac{x^2}{a_x^2+\xi}+\frac{y^2}{a_y^2+\xi}+\frac{z^2}{a_z^2+\xi}=1 \tag{7}$$

The external electrical potential given in Eq. (6) generates straightforwardly an external electric field and the corresponding density current. Results given in Eqs (5) and (6) are the electrostatics counterpart of the well known Eshelby theorems about the elastic behaviour of an ellipsoid embedded in a isotropic medium [21,22]. However, the electric version of these results can be applied for determining the field behaviour for a slit-crack and a circular crack in the following way.

2.1. Slit crack theory

We begin considering a slit crack as described in Fig. 1. So, in the previous equations we perform the limit of a_x diverging to infinity in order to simulate an elliptic cylinder aligned with the x-axis. We define e as the aspect ratio a_z/a_y and we define $b = a_y$. The limit of a_z approaching to zero (i.e. $e \to 0$), which mimics the flat shape of the crack, will be made in a second phase. Therefore, Eq. (6) with the condition $a_x \to \infty$, can be written in the following simplified form:

$$\varphi(x,y,z) = -E_{0x}x - E_{0y}y\frac{1-B(\xi)}{1-B(\infty)} - E_{0z}z\frac{1-A(\xi)}{1-A(\infty)} \tag{8}$$

where the functions $A(\xi)$ and $B(\xi)$ can be deduced from Eq. (6) and they have been defined as follows:

$$e=\frac{a_z}{a_y}, \quad b=a_y \quad \Rightarrow \quad \begin{cases} A(\xi) = \frac{a_ya_z}{2}\int\limits_0^{\xi}\frac{ds}{\sqrt{(s+a_z^2)^3\left(s+a_y^2\right)}} = \frac{e}{2}\int\limits_0^{\xi/b^2}\frac{d\eta}{\sqrt{(\eta+e^2)^3(\eta+1)}} \\ B(\xi) = \frac{a_ya_z}{2}\int\limits_0^{\xi}\frac{ds}{\sqrt{(s+a_z^2)\left(s+a_y^2\right)^3}} = \frac{e}{2}\int\limits_0^{\xi/b^2}\frac{d\eta}{\sqrt{(\eta+e^2)(\eta+1)^3}} \end{cases} \tag{9}$$

The integrals appearing in the previous relationships can be computed in closed form by means of standard substitutions [23], obtaining the results:

$$\begin{cases} A(\xi) = \frac{e}{e^2-1}\left\{\frac{\sqrt{1+\frac{\xi}{b^2}}}{\sqrt{e^2+\frac{\xi}{b^2}}} - \frac{1}{e}\right\} \Rightarrow A(\infty) = \frac{1}{e+1} \\ B(\xi) = \frac{e}{1-e^2}\left\{\frac{\sqrt{e^2+\frac{\xi}{b^2}}}{\sqrt{1+\frac{\xi}{b^2}}} - e\right\} \Rightarrow B(\infty) = \frac{e}{e+1} \end{cases} \tag{10}$$

We want to find a compact and explicit expression furnishing the electrical potential in the entire space. Therefore, we try to eliminate the variable ξ in relative expressions. From Eq. (7), performing the limit of $a_x \to \infty$, we obtain the following relation that allows us to find an explicit expression for the variable ξ:

$$\frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} = 1 \;\Rightarrow\; \frac{y^2}{1+\xi/b^2} + \frac{z^2}{e^2+\xi/b^2} = b^2 \tag{11}$$

By letting $\alpha = 1+\xi/b^2$ and $\beta = e^2+\xi/b^2$ we may write down the following system:

$$\begin{cases} \frac{y^2}{\alpha} + \frac{z^2}{\beta} = b^2 \\ \alpha - \beta = 1 - e^2 \end{cases} \tag{12}$$

As one can see in Eq. (10), we are interested in the following ratio, which can be derived from Eq. (12) by means of straightforward computations:

$$\frac{\alpha}{\beta} = \frac{1+\xi/b^2}{e^2+\xi/b^2} = \frac{z^2 - y^2 + b^2\left(1-e^2\right) + \sqrt{\left[z^2 - y^2 + b^2\left(1-e^2\right)\right]^2 + 4z^2y^2}}{2z^2} \tag{13}$$

Now, we may calculate the two relevant ratios appearing in Eq. (8) and their limiting values with e approaching zero:

$$\begin{cases} \frac{1-B(\xi)}{1-B(\infty)} = \frac{1}{1-e}\left[1 - e\sqrt{\beta/\alpha}\right] \overset{e\to 0}{\longrightarrow} 1 \\ \frac{1-A(\xi)}{1-A(\infty)} = \frac{1}{e-1}\left[e - \sqrt{\alpha/\beta}\right] \overset{e\to 0}{\longrightarrow} \left.\sqrt{\frac{\alpha}{\beta}}\right|_{e=0} \end{cases} \tag{14}$$

Summing up, the relation for the electric potential, in the space outside the crack, can be written in the following simplified version:

$$\varphi(x,y,z) = -E_{0x}x - E_{0y}y - E_{0z}z\left.\sqrt{\frac{\alpha}{\beta}}\right|_{e=0} \tag{15}$$

or, applying Eq. (13), in the final form:

$$\varphi(x,y,z) = -E_{0x}x - E_{0y}y - E_{0z}\frac{z}{|z|}\frac{1}{\sqrt{2}}\sqrt{z^2 - y^2 + b^2 + \sqrt{\left[z^2 - y^2 + b^2\right]^2 + 4z^2y^2}} \tag{16}$$

This expression furnishes the total electric potential in a region where a slit-crack obstructs the flow of a uniform current density. It is interesting to note that such a potential is a continuous function of the variables x, y and z in the entire space except for the region of the plane representing the crack.

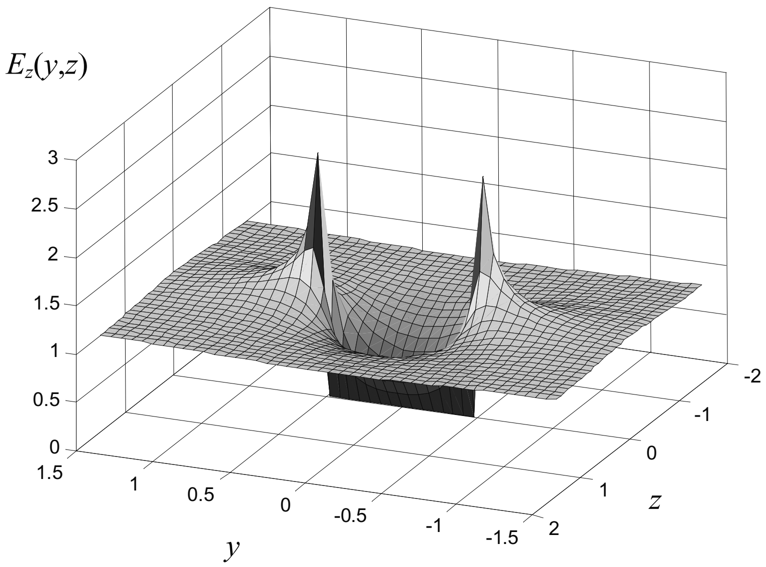

Fig. 5. Plot of the component E_z of the electric field around a slit-crack versus the geometrical variables y and z. One can observe the singular behaviour of the field near the crack tips. We have used the quantities $b = 0.5$ m and $E_{0z} = 1$ V/m.

In order to define a sort of field intensity factor, we consider a simpler case with $E_{0x} = E_{0y} = 0$. So, we may calculate the electric field along the z-axis, which is the main direction of propagation of the current density. For $z > 0$ we obtain:

$$E_z(y,z) = -\frac{\partial \varphi(y,z)}{\partial z} = E_{0z}\frac{1}{2\sqrt{2}\,|y|}\left[1 + \frac{z^2+y^2+b^2}{\sqrt{[z^2-y^2+b^2]^2+4z^2y^2}}\right]\cdot \cdot\sqrt{\sqrt{[z^2-y^2+b^2]^2+4z^2y^2}-z^2+y^2-b^2} \tag{17}$$

In Fig. 5 one can find the two-dimensional plot of the component of the electric field $E_z(y,z)$ versus the variables y and z obtained for a slit-crack with $b = 0.5$ and $E_{0z} = 1$. The two infinite peaks at the crack tips are evident. Moreover, by using Eq. (17), we may evaluate the intensity of the electric field on the plane $z = 0$ containing the crack:

$$\lim_{z\to 0} E_z(y,z) = \begin{cases} \frac{|y|E_{0z}}{\sqrt{y^2-b^2}} & \text{if} \quad y^2-b^2>0 \\ 0 & \text{if} \quad y^2-b^2<0 \end{cases} \tag{18}$$

It is evident that the electric field is zero inside the crack and it assumes singular behaviour in correspondence to the crack tips. Moreover the field become uniform when the distance from the crack is great. By letting $\eta = y - b$ (the distance from the crack) we may exactly define the electric field intensity factor as follows:

$$K = \lim_{\substack{\eta\to 0\\ z\to 0}} \sqrt{2\pi\eta}E_z(\eta,z) = \sqrt{b\pi}E_{0z} \tag{19}$$

2.2. Circular crack theory

Similar arguments can be applied to circular cracks (see Fig. 2 for the geometry). In this case the basic Eqs (6) and (7) should be applied under the condition $a_x = a_y = a$, where a is the radius of the crack. Moreover, we may define the aspect ratio $e = a_z/a_x = a_z/a_y$ and we will perform the limit $e \to 0$ in successive phases. The electric potential given by Eq. (6) may be written in a simplified form:

$$\varphi(x,y,z) = -(E_{0x}x + E_{0y}y)\frac{1-B(\xi)}{1-B(\infty)} - E_{0z}z\frac{1-A(\xi)}{1-A(\infty)} \tag{20}$$

where the functions $A(\xi)$ and $B(\xi)$ can be deduced from Eq. (6) and they have been defined as follows:

$$e = \frac{a_z}{a_x} = \frac{a_z}{a_y},\ a = a_x = a_y \ \Rightarrow\ \begin{cases} A(\xi) = \frac{a_x^2 a_z}{2}\int\limits_0^{\xi}\frac{ds}{(s+a_x^2)\sqrt{(s+a_z^2)^3}} = \frac{e}{2}\int\limits_0^{\xi/a^2}\frac{d\eta}{(\eta+1)\sqrt{(\eta+e^2)^3}} \\ B(\xi) = \frac{a_x^2 a_z}{2}\int\limits_0^{\xi}\frac{ds}{(s+a_x^2)^2\sqrt{s+a_z^2}} = \frac{e}{2}\int\limits_0^{\xi/a^2}\frac{d\eta}{(\eta+1)^2\sqrt{\eta+e^2}} \end{cases} \tag{21}$$

With the help of straightforward integration techniques [23] we have found the following complete solutions of the integrals appearing in Eq. (21):

$$\begin{cases} A(\xi) = \frac{e}{(1-e^2)^{3/2}}\left\{\frac{\sqrt{1-e^2}}{e} - \frac{\sqrt{1-e^2}}{\sqrt{e^2+\frac{\xi}{a^2}}} + \arctan\frac{e}{\sqrt{1-e^2}} - \arctan\frac{\sqrt{e^2+\frac{\xi}{a^2}}}{\sqrt{1-e^2}}\right\} \\ A(\infty) = \frac{e}{(1-e^2)^{3/2}}\left\{\frac{\sqrt{1-e^2}}{e} + \arctan\frac{e}{\sqrt{1-e^2}} - \frac{\pi}{2}\right\} \\ B(\xi) = \frac{e}{2(1-e^2)^{3/2}}\left\{\frac{\sqrt{1-e^2}\sqrt{e^2+\frac{\xi}{a^2}}}{1+\frac{\xi}{a^2}} - e\sqrt{1-e^2} - \arctan\frac{e}{\sqrt{1-e^2}} + \arctan\frac{\sqrt{e^2+\frac{\xi}{a^2}}}{\sqrt{1-e^2}}\right\} \\ B(\infty) = \frac{e}{2(1-e^2)^{3/2}}\left\{\frac{\pi}{2} - e\sqrt{1-e^2} - \arctan\frac{e}{\sqrt{1-e^2}}\right\} \end{cases} \tag{22}$$

Such expressions are useful to evaluate the limiting value (for e approaching zero) of the ratios appearing in Eq. (20):

$$\begin{cases} \frac{1-B(\xi)}{1-B(\infty)} \overset{e\to 0}{\longrightarrow} 1 \\ \frac{1-A(\xi)}{1-A(\infty)} \overset{e\to 0}{\longrightarrow} \frac{2}{\pi}\left[\sqrt{\frac{a^2}{\xi}} + \arctan\sqrt{\frac{\xi}{a^2}}\right] \end{cases} \tag{23}$$

As before, we want to eliminate the variable ξ in such a way to obtain the electric potential in orthogonal coordinates. For the circular crack Eq. (7) reduces to:

$$\frac{x^2}{a_x^2+\xi} + \frac{y^2}{a_y^2+\xi} + \frac{z^2}{a_z^2+\xi} = 1 \ \Rightarrow\ \frac{x^2+y^2}{1+\xi/a^2} + \frac{z^2}{e^2+\xi/a^2} = a^2 \tag{24}$$

In the limit of $e \to 0$ we obtain from Eq. (24):

$$\left.\frac{a^2}{\xi}\right|_{e=0} = \frac{\sqrt{(x^2+y^2+z^2-a^2)^2+4a^2z^2} - x^2 - y^2 - z^2 + a^2}{2z^2} \tag{25}$$

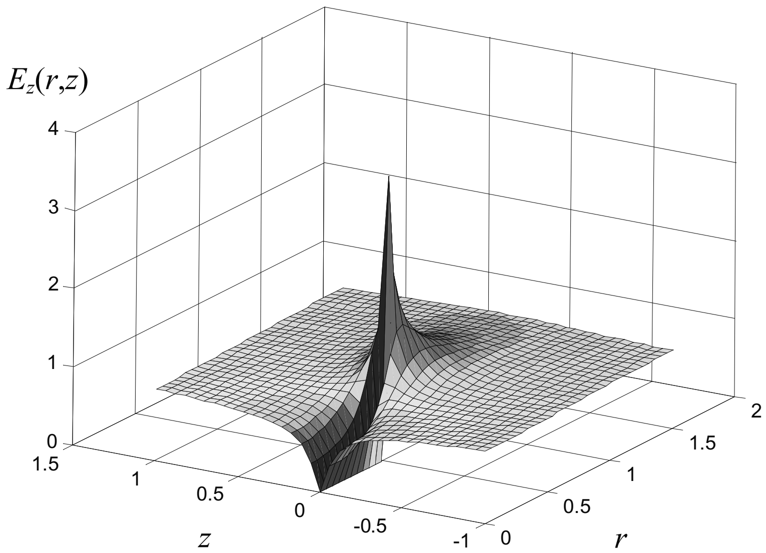

Fig. 6. Plot of the component E_z of the electric field around a circular crack versus the geometrical variables r and z. One can observe the singular behaviour of the field near the crack border. We have used the quantities $a = 0.5$ m and $E_{0z} = 1$ V/m.

So, substituting Eq. (25) in Eq. (23) and Eq. (23) in Eq. (20) we obtain the final solution for the electrical potential around a circular crack:

$$\varphi(x, y, z) = -E_{0x}x - E_{0y}y - E_{0z}z\frac{2}{\pi}\left[q + \arctan\frac{1}{q}\right] \tag{26}$$

$$\text{where} \quad q = \frac{1}{\sqrt{2}\,|z|}\sqrt{\sqrt{\left(x^2 + y^2 + z^2 - a^2\right)^2 + 4a^2z^2} - x^2 - y^2 - z^2 + a^2} \tag{27}$$

In order to define a sort of field intensity factor, we consider a simpler case with $E_{0x} = E_{0y} = 0$. So, we may calculate the electric field along the z-axis, which is the main direction of propagation of the current density:

$$E_z(r, z) = -\frac{\partial\varphi(r, z)}{\partial z} = E_{0z}\frac{2}{\pi}\left[q + \arctan\frac{1}{q} + z\frac{\partial q}{\partial z}\frac{q^2}{1 + q^2}\right], \quad r = \sqrt{x^2 + y^2} \tag{28}$$

Here, to simplify the results, we have defined the variable r because of the cylindrical symmetry of the problem. In Fig. 6 one can find the two-dimensional plot of the component of the electric field $E_z(r, z)$ versus the variables r and z obtained for a circular crack with $a = 0.5$ and $E_{0z} = 1$. The infinite peak at the crack border $r = a$ is evident. Finally, we may evaluate the intensity of the electric field on the plane $z = 0$ containing the crack:

$$\lim_{z\to 0} E_z(r, z) = \begin{cases} \frac{2E_{0z}}{\pi}\left[\frac{a}{\sqrt{r^2-a^2}} + \arctan\frac{\sqrt{r^2-a^2}}{a}\right] & \text{if} \quad r > a \\ 0 \quad\quad \text{if} \quad r < a \end{cases} \tag{29}$$

For the circular crack (see Fig. 2) the distance from the border of the crack is given by $\eta = r - a$ where a is the radius of the crack. Thus, the stress intensity factor is given by:

$$K = \lim_{\substack{\eta \to 0 \\ z \to 0}} \sqrt{2\pi\eta} E_z(\eta, z) = \frac{2\sqrt{a}}{\sqrt{\pi}} E_{0z} \tag{30}$$

2.3. Comparison with elastic results

It is interesting to observe that the singular behaviour of the electric filed near the border of the cracks, described by Eq. (19) for slit-cracks and by Eq. (30) for circular cracks, is absolutely similar to that obtained for the stress filed in elasticity as shown in Eqs (1) and (2).

3. Electric field density of state

In a pioneering paper [16] on the density of states, it is shown that for a crystal, under the assumption of harmonicity for the interatomic forces the frequency distribution function of elastic vibrations has analytic singularities. In this case, the nature of the singularities depends only on the number of dimensions of the crystal. In this section we study numerically the spatial fluctuations of the local electric field induced by a constant applied electric field in media containing cracks. It is found that the density of states for the electric field exhibits sharp peaks in the slope at certain critical points, which are analogous to van Hove singularities in the density of states for phonons in solids. It has been shown in earlier literature [14, 15] that the critical points are very prominent in dispersions with a regular, "crystal-like," structure. However, they disappear as the disorder increases. In our study we take into account a single crack in a solid conductor and we numerically evaluate the density of state for the electric field in a given region containing the crack.

3.1. Slit crack density of state

Firstly, we consider a slit-crack as shown in Fig. 1 and an applied electric field parallel to the z-axis. The resulting electric field is a two-dimensional field (on the plane $y - z$) described by Eq. (17) for $z > 0$ and by a similar expression for $z < 0$. So, we know the detailed mathematical behavior of the field on the plane under consideration. We define Σ as a region of the plane containing the crack (the segment $-b < y < b$, $z = 0$) having area A. The density of state for the electric field is defined by means of the following integral over the region Σ:

$$g(E) = \frac{1}{A} \iint_{\Sigma} \delta\left(E - \left|\bar{E}(\bar{r})\right|\right) dS \tag{31}$$

where δ is the Dirac delta function. We may say that the function $g(E)$ is defined in such a way that $g(E)dE$ is the total number of states in the range between E and $E + dE$, divided by the total area A of the region. The analytic evaluation of $g(E)$ has been made, for example, in Ref. [15] for a cylindrical inclusion. For a slit-crack, the complicated field distribution, given by Eq. (17), do not permit an analytical evaluation of the integral appearing in Eq. (31). So, we have made a numerical investigation to obtain $g(E)$. We have used an applied electric field $E_{0z} = 1$ V/m and a slit-crack with the semi aperture

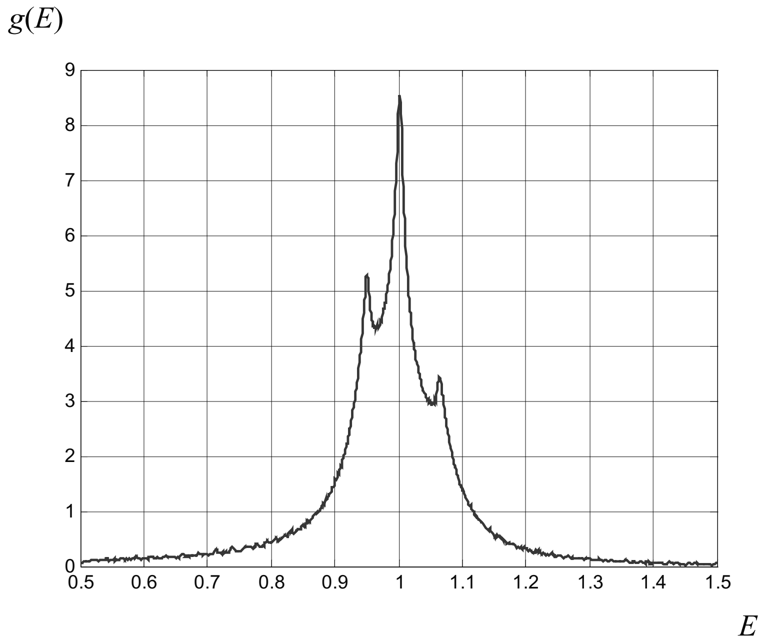

Fig. 7. Density of states $g(E)$ (see Eq. (31)) describing the distribution of the electric field in a planar region Σ ($-1.5 < y < 1.5$ and $-1.5 < z < 1.5$) containing a slit-crack with $b = 1$ m. The greatest peak corresponds to the value of the applied electric field $E_{0z} = 1$ V/m and other two peaks corresponding to van Hove singularities of the distribution of field.

$b = 1$ m. The region Σ has been assumed with a rectangular shape characterized by $-1.5 < y < 1.5$ and $-1.5 < z < 1.5$. The resulting density for the electric field is shown in Fig. 7. One can observe the greatest peak corresponding to the value of the applied electric field (1 V/m) and other two peaks corresponding to van Hove singularities of the distribution of field.

3.2. Circular crack density of state

The same procedure has been applied to the case of a circular crack. We have considered a circular crack with radius $a = 1$ m exposed to an electric field $E_{0z} = 1$ V/m. In this case we have taken as region Σ a rectangular part of the plane $r - z$: $0 < r < 1.5$ and $-1 < z < 1$. The resulting density of states is plotted in Fig. 8. Once again, one can observe the main peak corresponding to the bulk electric field $E_{0z} = 1$ V/m and two other peaks which represent the van Hove singularities in the distribution of the field. This means that, both for slit-cracks and for circular cracks, there are two particular values of the field, which appear much more frequently than all the other values in the electric field map.

Analysis of $g(E)$ and its van Hove singular points represents an interesting approach to quantify field fluctuations in complex media. The distribution of the local field is of fundamental and practical importance in understanding many crucial material properties such as breakdown phenomenon and the nonlinear behavior of composites. It is noteworthy that the density-of-states analysis of field fluctuations laid out in this paper for cracks can be applied to other field phenomena including strain fields in elastic media and velocity fields for flow through porous media.

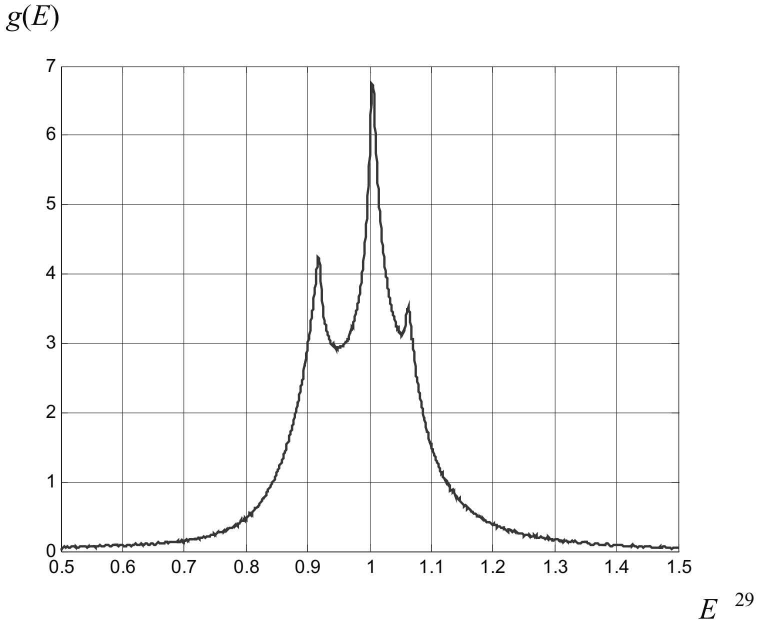

Fig. 8. Density of states $g(E)$ (see Eq. (31)) describing the distribution of the electric field in a planar region Σ ($0 < r < 1.5$ and $-1 < z < 1$) containing a circular crack with $a = 1$ m. The greatest peak corresponds to the value of the applied electric field $E_{0z} = 1$ V/m and other two peaks corresponding to van Hove singularities of the distribution of field.

4. Exponential laws for conductivity in micro-cracked conductors

The presence of cracks in a homogeneous conductor will influence the electrical properties of the medium. In this section we perform an analysis of the effects of the orientational distribution of cracks on the conductivity of an isotropic solid.

4.1. Population of slit cracks

Firstly, we consider a given distribution of slit-cracks as described in Fig. 3. Here, one can find the structure of the microcracked material with various degrees of order. The orientational distribution ranges from a situation where the cracks are parallel to a given direction to another one where they are random uniformly oriented in the space. We consider a given number N of randomly oriented cracks embedded in a homogeneous region (σ_1) of the plane $x - z$ with area A. To begin we consider a single crack lying on the $x - y$ plane and parallel to the x-axis (see Fig. 1). On the basis of the considerations reported in the first section of this paper, the following relations give the electrical field inside the ellipsoidal crack (see Eq. (5) with $\sigma_2 = 0$):

$$E_{i,k} = \frac{E_{0k}}{1 - L_k} \tag{32}$$

As before, for an elliptic cylinder aligned with the x-axis we assume the following hypotheses: we define e as the aspect ratio a_z/a_y and $b = a_y$. The limit of a_z approaching to zero (i.e. $e \to 0$), which mimics the flat shape of the crack, will be made in successive phases. The depolarising factors are given by:

$$L_x = 0, \quad L_y = \frac{e}{e+1}, \quad L_z = \frac{1}{e+1} \tag{33}$$

We treat the problem as a two-dimensional one in the $z-y$ plane. It means that the applied external field belongs to the $z-y$ plane. So, Eq. (32) combined with Eq. (33) furnishes the explicit relations for the internal electric field:

$$E_{i,y} = (e+1)\,E_{0y}, \quad E_{i,z} = \left(\frac{e+1}{e}\right)E_{0z} \tag{34}$$

Now, let's suppose that the crack section (ellipse in the plane $z-y$) is rotated around the x-axis of an angle θ (rotation in the $z-y$ plane). The tilting angle θ is measured starting from the positive semi axis x. By means of a straightforward rotation matrix we can write down the electric field inside the tilted ellipse, which mimics the shape of the crack:

$$\begin{cases} E_{i,y} = (e+1)\left(\cos^2\theta + \frac{1}{e}\sin^2\theta\right)E_{0y} + \cos\theta\sin\theta\frac{e^2-1}{e}E_{0z} \\ E_{i,z} = (e+1)\left(\frac{1}{e}\cos^2\theta + \sin^2\theta\right)E_{0z} + \cos\theta\sin\theta\frac{e^2-1}{e}E_{0y} \end{cases} \tag{35}$$

So, Eq. (35) furnishes the relationship between the external field and the internal one for a slit-crack with aspect ratio e after a rotation of an angle θ in the $z-y$ plane. We want to analyse the averaged effects of the orientational distribution of cracks and therefore we need to average Eq. (35) over all the possible orientation of a crack in the solid. Thus, the angle θ assumes, by hypotheses, the role of a random variable symmetrically distributed over the range $(-\pi/2, \pi/2)$. The symmetry of the probability density assures that the average value of $\sin(\theta)\cos(\theta)$ (appearing in Eq. (35)) is exactly zero and the result depends only on the average value of $\cos^2(\theta)$. So, we may define the following order parameter, which completely describes the state of order/disorder of the distribution of cracks:

$$P = \left\langle 1 - \cos(\theta)^2 \right\rangle \tag{36}$$

It is easy to observe that P assumes special values for particular angular distributions of cracks: if $P = 0$ all the cracks are parallel to the y-axes (horizontal order), if $P = 1$ all the cracks are parallel to the z-axes (vertical order) and if $P = 1/2$ the angle of rotation is uniformly distributed in the range $(-\pi/2, \pi/2)$ leading to a state of complete disorder (2D isotropic medium). The other values cover all the orientational distribution between the random and the parallel ones (see Fig. 3 for some examples). Finally, the averaged value of Eq. (9) is given by:

$$\begin{cases} \langle E_{i,y}\rangle = (e+1)\left(1 - P + \frac{P}{e}\right)E_{0y} \\ \langle E_{i,z}\rangle = (e+1)\left(\frac{1-P}{e} + P\right)E_{0z} \end{cases} \tag{37}$$

Now we may analyse the actual distribution of cracks: as above said, we consider a region of the plan $z-y$ having area A and N slit-cracks here dispersed with the angular distribution characterised by the order parameter P. Let c be the volume fraction of the embedded elliptic cylinders miming the cracks. It is given by $c = \pi a_z a_y N/A$ where a_z and a_y are linked by the relation $e = a_z/a_y$. The average value of the electrical field over the mixture is approximately given by:

$$\left\langle \overline{E} \right\rangle = (1-c)\,\overline{E_0} + c\left\langle \bar{E}_i \right\rangle \tag{38}$$

where we have considered the average electric field outside the inclusions approximately identical to the bulk field $\overline{E_0}$ (hypothesis of low cracks density). Then, we define $[\sigma]$ as the equivalent conductivity tensor of the whole mixture by means of the relation $\langle\bar{J}\rangle = [\sigma]\ \langle\bar{E}\rangle$ [23]; to evaluate $[\sigma]$ we may compute the average value of the current density vector inside the random material. We also define V as the total volume of the mixture, V_e as the total volume of the embedded ellipsoids and V_o as the volume of the remaining space among the cracks (so that $V = V_e \cup V_o$). The average value of $\bar{J}(\bar{r}) = \sigma(\bar{r})\,\bar{E}(\bar{r})$ is evaluated as follows:

$$\begin{aligned}\langle\overline{J}\rangle &= \frac{1}{V}\int_V \sigma(\bar{r})\,\bar{E}(\bar{r})d\bar{r} = \frac{1}{V}\sigma_1\int_{V_o}\bar{E}(\bar{r})d\bar{r} = \frac{1}{V}\sigma_1\int_{V_o}\bar{E}(\bar{r})d\bar{r} + \frac{1}{V}\sigma_1\int_{V_e}\bar{E}(\bar{r})d\bar{r} \\ &\quad -\frac{1}{V}\sigma_1\int_{V_e}\bar{E}(\bar{r})d\bar{r} = \varepsilon_1\left[\langle\bar{E}\rangle - c\,\langle\bar{E}_i\rangle\right]\end{aligned} \tag{39}$$

Note that $\langle\bar{J}\rangle$ and $\langle\bar{E}\rangle$ are not parallel vectors because of the presence of the average value of the internal electric field given by Eq. (37). Drawing a comparison between Eqs (37), (38) and (39) we may find complete expressions, which allows us to estimate the equivalent conductivity tensor $[\sigma]$:

$$\begin{cases}\sigma_\perp = \sigma_y = \sigma_1\frac{1-c}{1-c+c(e+1)\left(1-P+\frac{P}{e}\right)} \\ \sigma_{//} = \sigma_z = \sigma_1\frac{1-c}{1-c+c(e+1)\left(\frac{1-P}{e}+P\right)}\end{cases} \tag{40}$$

As above said, the volume fraction c is given by $c = \pi a_z a_y N/A$ or, remembering that $e = a_z/a_y$ and $b = a_y$, by $c = \pi b^2 eN/A$. Finally the limit for exactly flat cracks is obtained with $e \to 0$:

$$\begin{cases}\sigma_\perp = \sigma_y = \sigma_1\frac{1}{1+\frac{\pi Nb^2}{A}P} \cong \sigma_1\left[1-\frac{\pi Nb^2}{A}P\right] \\ \sigma_{//} = \sigma_z = \sigma_1\frac{1}{1+\frac{\pi Nb^2}{A}(1-P)} \cong \sigma_1\left[1-\frac{\pi Nb^2}{A}(1-P)\right]\end{cases} \tag{41}$$

We remember that Eq. (41) holds true only for low values of the cracks density. In fact, the sole approximation introduced in this procedure is contained in Eq. (31), which holds on only for low values of the cracks density. The first order expansions written in Eq. (41) are very useful to apply the iterated homogenisation method [24] that allows us to generalise the relations to higher values of the cracks density. The principles of this technique are here summarised: let's suppose that the effective conductivities of a microcracked medium are known to be $\sigma_\perp$ and $\sigma_{//}$. Now, if a small additional number of cracks ΔN is created in the matrix, the change in the elastic moduli is approximated to be that which arise if the same infinitesimal number of cracks were added to a uniform, homogeneous matrix with conductivities $\sigma_\perp$ and $\sigma_{//}$. This leads, when applied to Eq. (41), to the following difference equations:

$$\begin{cases}\sigma_\perp(N+\Delta N) = \sigma_\perp(N)\left[1-\frac{\pi b^2\Delta N}{A}P\right] \\ \sigma_{//}(N+\Delta N) = \sigma_{//}(N)\left[1-\frac{\pi b^2\Delta N}{A}(1-P)\right]\end{cases} \tag{42}$$

When the number of additional cracks ΔN assumes the role of an infinitesimal quantity the iterated homogenisation method converges to the differential effective medium theory [9,10] and the difference equations given in Eq. (42) became a pair of differential equations:

$$\begin{cases} \frac{d\sigma_{\perp}}{dN} = -\frac{\pi b^2}{A} P \sigma_{\perp} \\ \frac{d\sigma_{//}}{dN} = -\frac{\pi b^2}{A} (1-P)\, \sigma_{//} \end{cases} \tag{43}$$

They can be simply solved obtaining the final results for the characteristic conductivities of the micro-cracked solid:

$$\begin{cases} \sigma_{\perp} = \sigma_1 \exp\left(-\frac{\pi b^2 N}{A} P\right) \\ \sigma_{//} = \sigma_1 \exp\left(-\frac{\pi b^2 N}{A} (1-P)\right) \end{cases} \tag{44}$$

So, we have obtained two exponential laws for the principal conductivities of a microcracked solid where slit-cracks have been dispersed following a statistical orientation described by the order parameter P. Such conductivities depend exponentially both on the crack size (half-length b) and the cracks density (ratio N/A).

4.2. Population of circular cracks

In the second part of this section we are dealing with three-dimensional distributions of circular planar cracks in isotropic solids (N cracks dispersed in a region with volume V). As, before the main feature of this analysis is given by the pseudo random orientation of the cracks inside the solid (see Fig. 4 for details). We consider a given orthonormal reference frame and we take as preferential direction of alignment the z-axis. Each crack embedded in the matrix is not completely random oriented. The overall medium has a positional disorder but a partial orientational order and it exhibits a uniaxial behaviour. The orientation of a crack is described by the following statistical rule: the principal axis (the normal direction) of each crack forms with the z-axis an angle θ, which follows a given probability density, $f(\theta)$ defined in $[0\ \pi]$ (see Fig. 2). The orientation of each crack is statistically independent from the orientation of the other ones. If $f(\theta) = \delta(\theta)$ (where δ is the Dirac delta function) we have all the cracks with $\theta = 0$ and therefore they are all oriented with the z-axis. If $f(\theta) = (1/2)\sin\theta$ all the cracks are uniformly random oriented in the space over all the possible directions. Any other statistical distributions $f\ (\theta)$ defines a transversely isotropic (uniaxial) material. In this section we develop a complete analysis of the effects of the state of order/disorder. This analysis allows us to evaluate the overall electric properties of the microcracked material.

$$\overline{E_i} = \frac{\left(\overline{E_0} \cdot \overline{n_x}\right) \overline{n_x}}{1-L_x} + \frac{\left(\overline{E_0} \cdot \overline{n_y}\right) \overline{n_y}}{1-L_y} + \frac{\left(\overline{E_0} \cdot \overline{n_z}\right) \overline{n_z}}{1-L_z} \tag{45}$$

This result simply derives from the sum of the three contributions to the electrical field along each axes. This expression may be written in explicit form (component by component), as follows:

$$E_{i,q} = \sum_{k}^{x,y,z} E_{0,k} \sum_{j}^{x,y,z} \frac{n_{j,k} n_{j,q}}{1-L_j} \tag{46}$$

where $n_{j,k}$ is the k-th component of the unit vector $\overline{n_j}$ ($j = x,\ y,\ z$). From now on, we are interested in the behaviour of an ellipsoid of revolution and therefore we use the simplified notation $L_x = L_y = L$ and $L_z = 1 - 2L$. It means that L is the depolarisation factor along the unit vectors $\overline{n_x}$ and $\overline{n_y}$ and 1-2L

is the depolarising vector along the axis $\overline{n_z}$. We may use spherical coordinates ψ, φ and θ to write down explicit expressions for the unit vectors:

$$\begin{cases} \overline{n_x} = (\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta, -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta, \sin\psi\sin\theta) \\ \overline{n_y} = (\sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta, -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta, -\cos\psi\sin\theta) \\ \overline{n_z} = (\sin\varphi\sin\theta, \sin\theta\cos\varphi, \cos\theta) \end{cases} \tag{47}$$

For the following derivations, we are interested in the average value of the electrical field inside the ellipsoid over the possible orientations of the ellipsoid itself and then we have to compute the average value of the quantity $n_{j,k}n_{j,q}$. The two angles ψ and φ are statistically independent from each other and distributed following a uniform probability density in the range [0 2π]. Performing the integration over the unit sphere, by means of spherical coordinates, we obtain, after some straightforward computations, the first step of the averaging procedure:

$$\begin{cases} \langle n_{x,x}n_{x,x}\rangle_{\psi,\varphi} = \langle n_{x,y}n_{x,y}\rangle_{\psi,\varphi} = \langle n_{y,x}n_{y,x}\rangle_{\psi,\varphi} = \langle n_{y,y}n_{y,y}\rangle_{\psi,\varphi} = \frac{1}{4}\left(1+\cos^2\theta\right) \\ \langle n_{x,z}n_{x,z}\rangle_{\psi,\varphi} = \langle n_{y,z}n_{y,z}\rangle_{\psi,\varphi} = \langle n_{z,x}n_{z,x}\rangle_{\psi,\varphi} = \langle n_{z,y}n_{z,y}\rangle_{\psi,\varphi} = \frac{1}{4}\left(1-\cos^2\theta\right) \\ \langle n_{z,z}n_{z,z}\rangle_{\psi,\varphi} = \cos^2\theta \end{cases} \tag{48}$$

Here, the symbol $\langle\rangle_{\psi,\varphi}$ represents the average value over the angles ψ and φ. The terms that not appear in the previous Eq. (48) are all zero. The angle θ is statistically independent from the others and distributed following an arbitrary probability density $f(\theta)$, which defines the degree of ordering of the medium, ranging from perfect order ($f(\theta) = \delta(\theta)$), to complete disorder ($f(\theta) = (1/2)\sin\theta$). The statistical distribution of the angle θ is well described by the following order parameter S, which takes into account the average value of the second Legendre polynomial:

$$S = \langle P_2(\cos\theta)\rangle_\theta = \left\langle \frac{3}{2}\cos^2(\theta) - \frac{1}{2} \right\rangle_\theta = \int_0^\pi \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right) f(\theta) d\theta \tag{49}$$

where θ is the angle that the particle (its unit vector $\overline{n_z}$) makes with the preferential direction given by the axis z of the main reference frame (the symbol $\langle\rangle_\theta$ represents the average value over the angle θ). By means of the definition of such order parameter we may perform the final averaging over the tilting angle θ:

$$\begin{cases} \langle n_{x,x}n_{x,x}\rangle_{\psi,\varphi,\theta} = \langle n_{x,y}n_{x,y}\rangle_{\psi,\varphi,\theta} = \langle n_{y,x}n_{y,x}\rangle_{\psi,\varphi,\theta} = \langle n_{y,y}n_{y,y}\rangle_{\psi,\varphi,\theta} = \frac{2}{3}(S+2) \\ \langle n_{x,z}n_{x,z}\rangle_{\psi,\varphi,\theta} = \langle n_{y,z}n_{y,z}\rangle_{\psi,\varphi,\theta} = \langle n_{z,x}n_{z,x}\rangle_{\psi,\varphi,\theta} = \langle n_{z,y}n_{z,y}\rangle_{\psi,\varphi,\theta} = \frac{2}{3}(1-S) \\ \langle n_{z,z}n_{z,z}\rangle_{\psi,\varphi,\theta} = \frac{2S+1}{3} \end{cases} \tag{50}$$

Here, the symbol $\langle\rangle_{\psi,\varphi,\theta}$ represents the average value over the angles ψ, φ and θ; for sake of simplicity, from now on the indication of the angles on which the averaging is performed will be omitted. Therefore, the average value of the electrical field (inside the randomly oriented inclusion), given by Eq. (46), may be written as:

$$\begin{aligned} \langle E_{i,x}\rangle &= \frac{E_{0,x}}{3}\left[\frac{S+2}{1-L} + \frac{1-S}{2L}\right], \ \langle E_{i,y}\rangle = \frac{E_{0,y}}{3}\left[\frac{S+2}{1-L} + \frac{1-S}{2L}\right], \\ \langle E_{i,z}\rangle &= \frac{E_{0,z}}{3}\left[\frac{2(1-S)}{1-L} + \frac{2S+1}{2L}\right] \end{aligned} \tag{51}$$

In Fig. 4 one can find the structure of the microcracked material with various degrees of order: we consider a given number of randomly oriented cracks embedded in a homogeneous matrix (σ_1). As before, let $c$be the volume fraction of the embedded ellipsoids. Equations (38) and (39) hold true also in this three-dimensional case:

$$\left\langle \overline{E} \right\rangle = (1-c)\,\overline{E_0} + c\left\langle \bar{E}_i \right\rangle \tag{52}$$

$$\left\langle \overline{J} \right\rangle = \varepsilon_1\left[\left\langle \bar{E} \right\rangle - c\left\langle \bar{E}_i \right\rangle\right] \tag{53}$$

Drawing a comparison between Eqs (51),(52) and (53) we may find complete expressions, which allows us to estimate the equivalent conductivity tensor $[\sigma]$, defined by means of the relation $\left\langle \bar{J} \right\rangle = [\sigma]\ \left\langle \bar{E} \right\rangle$:

$$[\sigma]\ = \begin{bmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_{//} \end{bmatrix} \tag{54}$$

where the longitudinal and transversal conductivities are given by:

$$\begin{cases} \sigma_\perp = \sigma_1 \frac{1-c}{1-c+\frac{c}{3}\left[\frac{S+2}{1-L}+\frac{1-S}{2L}\right]} \\ \sigma_{//} = \sigma_1 \frac{1-c}{1-c+\frac{c}{3}\left[\frac{2(1-S)}{1-L}+\frac{2S+1}{2L}\right]} \end{cases} \tag{55}$$

The depolarisation factor L may be computed in closed form and the result depends on the shape of the ellipsoid; if it is oblate ($e = a_z/a_x = a_z/a_y < 1$) [16,23]:

$$L = \frac{e}{4\left(\sqrt{1-e^2}\right)^3}\left[\pi - 2e\sqrt{1-e^2} - 2\mathrm{arctan}\frac{e}{\sqrt{1-e^2}}\right] \cong \frac{\pi\, e}{4} \quad (e << 1) \tag{56}$$

With the aim of modelling a circular crack we will use the limit of $e \to 0$ (strongly oblate ellipsoid).

Here the volume fraction c is given by $c = 4\pi a_x^2 a_z N/(3V)$ or, remembering that $e = a_z/a_x = a_z/a_y$ and $a = a_x = a_y$, by $c = 4\pi a^3 eN/\,(3V)$. The limit for exactly flat cracks is obtained with $e \to 0$. By substituting Eq. (56) in Eq. (55) and performing such a limit we obtain:

$$\begin{cases} \sigma_\perp = \sigma_1 \frac{1}{1+\frac{8}{9}\frac{a^3N}{V}(1-S)} \cong \sigma_1\left[1-\frac{8}{9}\frac{a^3N}{V}\,(1-S)\right] \\ \sigma_{//} = \sigma_1 \frac{1}{1+\frac{8}{9}\frac{a^3N}{V}(2S+1)} \cong \sigma_1\left[1-\frac{8}{9}\frac{a^3N}{V}\,(2S+1)\right] \end{cases} \tag{57}$$

So, we have obtained final expressions for the principal conductivities of a conductor where circular cracks are distributed following a statistical orientation given by the order parameter S. They are correct only under the assumption of low density of cracks. In Eq. (57) we have also shown the first order expansions that are valid for very low cracks density. They are useful to apply a differential procedure similar to that described in the case of slit-cracks. This procedure leads to a couple of differential equations that can be solved in closed form obtaining the final results for the principal conductivities, which should be valid with any value of the density of cracks:

$$\begin{cases} \sigma_\perp = \sigma_1 \exp\left(-\frac{8a^3N}{9V}\,(1-S)\right) \\ \sigma_{//} = \sigma_1 \exp\left(-\frac{8a^3N}{9V}\,(2S+1)\right) \end{cases} \tag{58}$$

It is interesting to observe that our solutions (given by Eq. (44) for slit cracks and by Eq. (58) for circular cracks) depend exponentially on the cracks density (N/A in two-dimensional cases or N/V in three-dimensional ones). This fact explains the very speed reduction of the conductivity of a medium with an increasing number of cracks in a given region.

5. Conclusions

We have analytically evaluated the effects of cracks in solid conductors. Firstly we have found the exact expressions for the electric potentials and fields in a region where a crack is present. A uniform density current flowing in the conductor strongly modifies its uniformity in the region near the crack, generating singular behaviour of the electric field around the crack tips. The analysis has been carried out both for slit-cracks in two-dimensional electrostatics and for circular cracks in three-dimensional electrostatics. Such a study has conducted to the definition of the field intensity factor, which is analogue to the well-known stress intensity factor in continuum elasticity and fracture mechanics. So, the fluctuations of the field around a crack has been analysed by means of the so-called density of states of the field. We have numerically found that such a distribution exhibits sharp peaks very similar to the van Hove singularities of the density of states for phonons in solid. Finally, we have estimated the effective conductivities of microcracked solid conductors in terms of the orientational distribution of cracks in the materials. In particular we have shown that the statistical orientational distribution of cracks can me taken into account by means of suitable order parameters. The results for the microcracked conductors are given by exponential laws that well describe the strong reduction of the conductivity of a medium with an increasing number of cracks in a given region.

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