Effects of the Orientational Distribution of Cracks in Solids

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We derive a theory for the elastic characterization of multicracked solids based on a homogenization technique. We consider a material containing a two-dimensional arbitrary distribution of parallel slit cracks which is elastically equivalent to a crystal with orthorhombic symmetry. We obtain explicit expressions for the macroscopic elastic stiffness tensor which is found to depend upon both the density of cracks and their angular distribution, here described by a suitable order parameter. For the isotropic case, we find that the degradation depends exponentially on the crack density. In addition, we show an unusual elastic behavior of a multicracked medium in the plane strain condition: for a negative Poisson ratio, we obtain an effective Young modulus greater than the actual value of the host matrix.

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The macroscopic degradation of brittle materials is governed by the generation of cracks and by their mutual interactions. While linear elastic fracture mechanics (LEFM) provides the basic understanding of the failure instability for a single crack, the overall mechanical behavior actually depends upon the positional and orientational distribution of an assembly of cracks.

The basic result of LEFM is represented by the Griffith theory [1]: upon loading, a single crack with length $l > l_c$ will further grow (eventually producing materials failure), while if $l < l_c$, it will remain stable. Here the critical length l_c is inversely proportional to the square of the tensile stress applied to the material. The Griffith criterion has been extensively verified in glass specimens containing cracks of controlled length [2], and recently validated by atomistic simulations in an ideally brittle, single-crystal system [3].

When considering the overall materials behavior, a key conceptual issue consists in evaluating the effective elastic properties (e.g., the stiffness tensor) that determine the mechanical performance of the system containing a given distribution of cracks. Under this respect, the characterization of multicracked materials belongs to the vast field of homogenization techniques [4,5]. The most extensively studied elastic homogenization theory is basically addressed to a dilute dispersion of spherical [6] or ellipsoidal [7] inclusions into a solid matrix. The results valid for a dilute assembly of particles (e.g., defects or inclusions) have been generalized in several ways, including the iterated homogenization technique [8] and the differential effective medium theory [9,10].

The predictions of homogenization techniques are of paramount importance in several fields, ranging from materials science, to geophysics, to biology. Complex materials with negative Poisson ratio [11] or composite materials with negative stiffness phases [12] have been in fact investigated by such theoretical devices. On the other hand, interesting applications have been developed for rocks, where cracking is originated by a number of geological processes, like thermal gradients and tectonic stresses. Experiments on temperature- or stress-induced cracking suggest that the former process produces a fairly isotropic distribution of predominantly intergranular cracks, while the latter generates a strongly anisotropic distribution of intragranular and transgranular cracks, with the majority of cracks oriented parallel to the direction of the maximum principal stress [13,14]. In addition, homogenization techniques have been applied to investigate the nonisotropic fracture mechanisms in mineralized biological tissues, like bone and dentin [15]. The site-specific accumulation of cracks has been studied as well, since it is considered a key factor affecting the crack resistance of bones [16].

This scenario stands for the conceptual framework underlying the present investigation, which is addressed to better understanding the effects of cracks on the mechanical properties of an elastic matrix by means of homogenization methodologies. First, we observe that a solid body with a two-dimensional arbitrary orientational distribution of parallel slit cracks is ultimately equivalent to a crystal with orthorhombic symmetry, and then we explicitly derive its stiffness tensor. Moreover, we demonstrate that the angular distribution of cracks can be described by a sole order parameter, which takes into account all of the microscopic features reflected macroscopically. Finally, by developing an iterated homogenization procedure, we show that the Young modulus of such a solid exponentially decays with the density of cracks.

The elementary object of our model is an ellipsoidal void with an infinitesimally thick minor axis, so as to mimic the flat shape of a crack. Treating the crack as a void and oblate ellipsoid of vanishing eccentricity is very convenient since we develop our arguments by taking profit from general results holding for ellipsoidal inclusions [17]. We consider an ellipsoid with semiaxes a_x, a_y, and a_z ($a_x > a_y > a_z > 0$) aligned, respectively, along the

x, y, and z axes of a given Cartesian frame of reference. If one of the principal axes of the ellipsoidal void, say a_x, becomes very large and the minor axis a_z becomes negligibly small, then the ellipsoid reduces to a slitlike crack with half-length $a_y = b$ (see Fig. 1).

Hence, we considered a void with infinite a_x and finite a_y and a_z. In other words, the void is an elliptic cylinder aligned with the x-axes. We define the aspect ratio as $e = a_z/a_y$. The actual slitlike crack geometry is then described by the limit $a_z \rightarrow 0$ or, equivalently, $e \rightarrow 0$. The present Letter is addressed to investigating a multicracked solid with a given distribution of cracks. Therefore, a number of slit-cracks are randomly placed within the plane z-y. We may consider different cases as described in Fig. 2, ranging from distributions with cracks parallel to a given direction [Figs. 2(a) and 2(d)], to cracks uniformly oriented in space [Fig. 2(c)], to cracks preferentially oriented in some direction [Fig. 2(b)]. This latter configuration represents the intermediate distribution between the liming cases of full order [Figs. 2(a) and 2(d)] and complete disorder [Fig. 2(c)]. The angular distribution of cracks is described by an order parameter P (see below), while the host solid matrix is characterized by the Young modulus E and by the Poisson ratio ν. The. present theory holds for any $E > 0$ and $-1 < \nu < 1/2$.

Let us consider at first the case of a single crack created into the host solid which, in turn, was previously accommodated in a state of constant elastic strain due to some external load. The resulting state of strain of the cracked solid is well described by the Eshelby theory [18]. In its most general formulation, the Eshelby theory provides the elastic behavior of an ellipsoidal inclusion embedded into a matrix. The most important feature is that the internal stress and strain fields are constant if the external strain is constant. Such internal elastic fields can be calculated from the external applied ones by means of the so-called Eshelby tensor [19]. It depends only on geometrical factors of the ellipsoidal inclusion and on the Poisson ratio ν of the host matrix. So, the Eshelby tensor contains all the physical information needed to predict the mechanical interaction between the inclusion and the matrix under external load. For voids or cavities, the relationship between the original applied strain $\boldsymbol{\varepsilon}_0$ and the induced internal strain $\boldsymbol{\varepsilon}_i$ is given by $\boldsymbol{\varepsilon}_i = \{\mathbf{I} - \mathbf{S}\}^{-1}\boldsymbol{\varepsilon}_0$ where $\mathbf{I}$ is the identity tensor and $\mathbf{S}$ is the Eshelby tensor. If we consider a void shaped as the elliptic cylinder described above, the Eshelby tensor becomes simply dependent on the aspect ratio $e = a_z/a_y$ and on the Poisson ratio ν of the matrix [19]. We used this result for a single crack as defined in Fig. 1, although arbitrarily rotated around the x-axis of an angle θ. The angle θ has the role of a random variable symmetrically distributed over the range $(-\pi/2, \pi/2)$. We now define the order parameter $P = \langle \sin^2\theta \rangle$ which completely describes the state of order ($\theta = 0$ or $\pi/2$) and disorder ($0 < \theta < \pi/2$) of the distribution of cracks. It is easy to recognize that P assumes special values for particular angular distributions of cracks as indicated in Fig. 2. So, we computed the average value (hereafter indicated by $\langle \cdot \rangle$ squares) of the internal strain $\langle \boldsymbol{\varepsilon}_i \rangle$ over all the possible orientations: $\langle \boldsymbol{\varepsilon}_i \rangle = \langle \{\mathbf{I} - \mathbf{S}\}^{-1}\boldsymbol{\varepsilon}_0 \rangle = \langle \{\mathbf{I} - \mathbf{S}\}^{-1} \rangle \boldsymbol{\varepsilon}_0 = \mathbf{C}\boldsymbol{\varepsilon}_0$, where $\mathbf{C}$ is the averaged Wu's tensor [5] depending only on e, ν, and P.

We move now to the case of an actual distribution of several cracks: we consider a region of the plane z-y having area A and containing N elliptic inclusions with aspect ratio e (equivalent to slit-cracks in the limit $e \rightarrow 0$), and an angular distribution characterized by the order parameter P (see Fig. 2). The volume fraction c of the inclusions is given by N times the area $\pi a_z a_y$ of the elliptic base, divided by total area A of the region of interest, so that $c = \pi a_z a_y N/A$. Since $a_y = b$ and $e = a_z/a_y$, we easily obtain

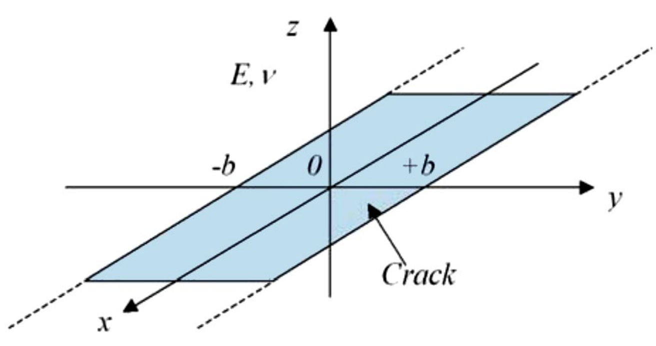

FIG. 1 (color online). Geometrical representation of a slit-crack aligned along the x-axis. The half-length of the crack is b. The host matrix has Young modulus E and Poisson's ratio ν.

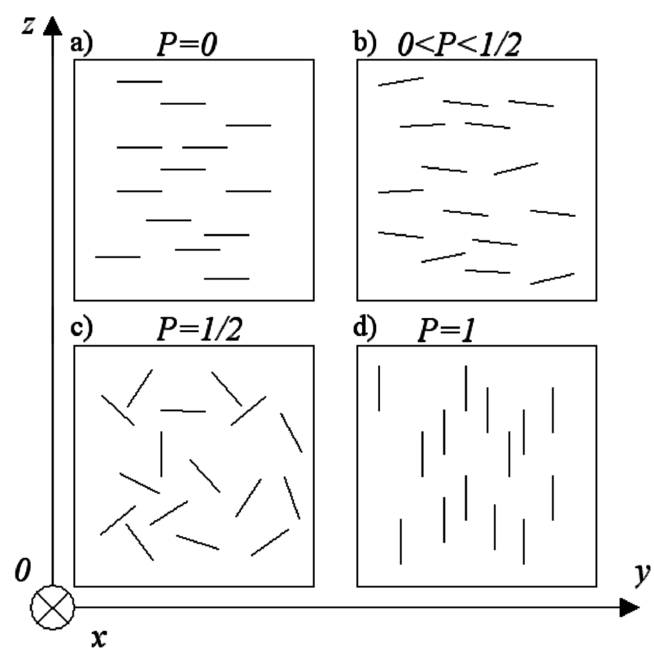

FIG. 2. Structure of a multicracked solid with slit-cracks along the x-axis. The order parameter P clearly indicates the state of order [panels (a) and (d)] or disorder [panel (c)].

the result $c = \pi e b^2 N/A$. Moreover, we can define the new characteristic quantity $\alpha = \pi b^2 N/A$: it is dimensionless, and it effectively represents the crack density. Hence, we may write $c = \alpha e$.

We need now to compute the average value of the strain $\langle \boldsymbol{\varepsilon} \rangle$ and of the stress $\langle \mathbf{T} \rangle$ over the whole region of interest (i.e., area A). To begin, we work under the hypothesis of low cracks density so that the cracks are not interacting with each other. Therefore, we approximate the average value of the strain outside the cracks with the external applied strain $\boldsymbol{\varepsilon}_0$ (this is strictly true for a single crack exposed to a given external load), and we get $\langle \boldsymbol{\varepsilon} \rangle = c\langle \boldsymbol{\varepsilon}_i \rangle + (1-c)\boldsymbol{\varepsilon}_0$. By recalling the definition of the tensor $\mathbf{C}$, we finally obtain the sole approximated relation introduced in the present model, namely

$$\langle \boldsymbol{\varepsilon} \rangle = [(1-c)\mathbf{I} + c\mathbf{C}]\boldsymbol{\varepsilon}_0 \tag{1}$$

On the other hand, an exact result can be obtained for the stress

$$\langle \mathbf{T} \rangle = \mathbf{L}[\langle \boldsymbol{\varepsilon} \rangle - c\mathbf{C}\boldsymbol{\varepsilon}_0] \tag{2}$$

where $\mathbf{L}$ is the stiffness tensor of the homogeneous (i.e., noncracked) matrix and $\langle \boldsymbol{\varepsilon} \rangle$ is given by Eq. (1). We define the effective stiffness tensor $\mathbf{L}_{\text{eff}}$ of the cracked body through the relation $\langle \mathbf{T} \rangle = \mathbf{L}_{\text{eff}} \langle \boldsymbol{\varepsilon} \rangle$ [5,6]. By means of Eqs. (1) and (2), we obtain $\mathbf{L}_{\text{eff}} = \mathbf{L}\{\mathbf{I} - \mathbf{G}[\mathbf{I} + \mathbf{G}]^{-1}\}$, where $\mathbf{G} = \lim_{e\to 0} c\mathbf{C}$ and it depends on α, ν, and P. This crucial step is performed under the limiting condition $e \to 0$ in order to take into account the actual planar geometry of the cracks.

It is evident by the micro-geometry of the system that the solid is elastically anisotropic: along the x-axes, we find the alignment of slit-cracks, while along the y and z axes, we get different elastic behaviors because of the given orientation of the cracks. However, the three different behaviors along the three axes lead to an orthorhombic anisotropy for the whole system, so that

$$\mathbf{L}_{\text{eff}} = \begin{bmatrix} L_{1111} & L_{1122} & L_{3311} & 0 & 0 & 0 \\ L_{1122} & L_{2222} & L_{2233} & 0 & 0 & 0 \\ L_{3311} & L_{2233} & L_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{3131} \end{bmatrix} \tag{3}$$

where the relation $\langle T_{ij} \rangle = L_{ijkl} \langle \varepsilon_{kl} \rangle$ characterizes the multicracked solid. Here, L_{ijkh} are the entries of $\mathbf{L}_{\text{eff}}$ (corresponding to nine independent parameters). We can write the closed form expressions for the stiffness tensor entries in terms of the E, ν as well as in terms of the parameters P and α described above:

$$\begin{aligned} L_{1111} &= [4\alpha^2 P(1-P)(1+\nu)(1-\nu)^2 \\ &\quad + 2(1-\nu)\alpha + 1 - \nu]D^{-1}E \\ L_{2222} &= (1-\nu)[1 + 2(1-P)\alpha]D^{-1}E \\ L_{3333} &= (1-\nu)[1 + 2\alpha P]D^{-1}E \end{aligned} \tag{4}$$

$$\begin{gathered} L_{1122} = \nu[1 + 2(1-P)(1-\nu)\alpha]D^{-1}E \\ L_{2233} = \nu D^{-1}E \qquad L_{3311} = \nu[1 + 2P(1-\nu)\alpha]D^{-1}E \end{gathered} \tag{5}$$

$$\begin{aligned} L_{1212} &= \{(1+\alpha P)(1+\nu)\}^{-1}E \\ L_{2323} &= \{[1 + (1-\nu)\alpha](1+\nu)\}^{-1}E \\ L_{3131} &= \{[1 + \alpha(1-P)](1+\nu)\}^{-1}E \end{aligned} \tag{6}$$

where $D = [4\alpha^2 P(1-P)(1-\nu)^2 + 2(1-\nu)^2\alpha + 1 - 2\nu](1+\nu)$.

Equations (4)–(6) represent the complete characterization of a solid with a given distribution of slit-cracks under the sole hypothesis of low cracks density.

When the slit-cracks are uniformly random oriented in the y-z plane, the overall multicracked material is transversely isotropic (along the x-axis) and the corresponding stiffness tensor is given by Eqs. (4)–(6) with $P = 1/2$. As it is well known, the number of independent entries in the stiffness tensor decreases from nine to five moving from orthorhombic anisotropy to transverse isotropy. Typically, in two-dimensional elasticity, a transversely isotropic medium may be used under the conditions of plane stress or plane strain (and it appears as an isotropic material with effective moduli E_{eff} and ν_{eff}). Starting from Eqs. (4)–(6) (with $P = 1/2$), we may apply the iterated homogenization method [8] that allows us to remove the hypothesis of low cracks density. The principles of this technique are here summarized: let us suppose that the effective moduli of a multicracked medium are known to be E_{eff} and ν_{eff}. Therefore, if a small number of cracks ΔN is added to the matrix, the change in the elastic moduli is the same as in a uniform, homogeneous matrix with moduli E_{eff} and ν_{eff}. When the number of additional cracks ΔN assumes the role of an infinitesimal quantity, the iterated homogenization method converges to the differential effective medium theory [9,10]. Eventually, this approach leads to the following results under plane stress conditions:

$$E_{\text{eff}} = \frac{E}{\sqrt{\nu^2 + (1-\nu^2)e^{2\alpha}}} \qquad \nu_{\text{eff}} = \frac{\nu}{\sqrt{\nu^2 + (1-\nu^2)e^{2\alpha}}} \tag{7}$$

A similar procedure can be followed for the plane strain case, obtaining the effective elastic moduli as follows:

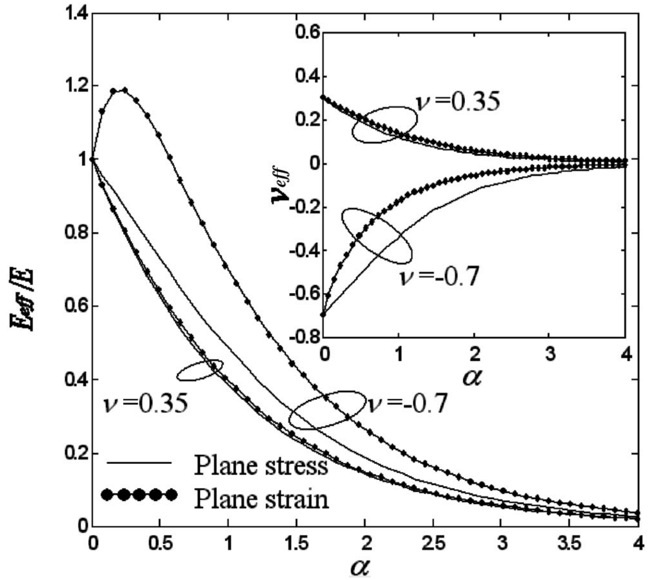

FIG. 3. Effective Young modulus and Poisson ratio for a multicracked solid under plane stress and plane strain conditions. The plots have been derived for two different homogeneous matrices having $\nu = 0.35$ and $\nu = -0.7$.

$$E_{\text{eff}} = E\frac{2\nu + (1-\nu)e^{\alpha}}{[\nu + (1-\nu)e^{\alpha}]^2(1+\nu)}$$
$$\nu_{\text{eff}} = \frac{\nu}{\nu + (1-\nu)e^{\alpha}} \tag{8}$$

It is important to observe that our solutions (given by Eq. (7) for plane stress and by Eq. (8) for plane strain) depend exponentially on the cracks density. In Fig. 3, these results have been represented versus the parameter α. A comparison between the plane stress and the plane strain cases has been drawn both for positive and negative Poisson's ratio.

Other approximations based on differential schemes are found in Ref. [20], leading to a similar exponential dependence. However, only the present theoretical device is properly suited to fully characterize the effective mechanical behavior observed in the two different loading conditions. In fact, an interesting unconventional behavior of the effective Young modulus of the multicracked solid has been found for negative Poisson ratio [11] under plane strain condition. When $-1 < \nu < -1/2$, we obtain, for low values of α, an effective Young modulus greater than the Young modulus of the original elastic matrix. More precisely, the effective Young modulus in such a case has a maximum for $\alpha = \ln[-3\nu/(1-\nu)]$ which can be approximated as $\alpha = -(1+2\nu)/(1-\nu)$, when α is small enough. This effect is shown in Fig. 3, where a value $\nu = -0.7$ is assumed. Our choice corresponds to the realistic case reported for foams with ν as small as -0.8 [11]. This effect is not present under plane stress conditions. The unusual behavior observed in plane strain conditions can be attributed to the specific meaning of the Young modulus in such a case: the elastically loaded plain-strain system has fewer degrees of freedom than the system in plane-stress because of the peculiar boundary conditions needed to avoid the appearing of out-of-plane strain in the solid. This means that, when one measures the Young modulus on a given direction in plane strain conditions, some other forces must be applied in the orthogonal directions in order to fulfill the plain strain state, generating a very special set of loading.

In conclusion, we have proved that the effective Young modulus of a multicracked solid exponentially decreases with increasing density of cracks. Moreover, for a negative Poisson ratio, a fractured medium can be stiffer than the original matrix under plane strain condition, contrary to common expectations.

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[1] A. A. Griffith, Phil. Trans. R. Soc. A **221**, 163 (1921).
[2] B.R. Lawn, *Fracture of Brittle Solids* (Cambridge University Press, Cambridge, 1975).
[3] A. Mattoni, L. Colombo, and F. Cleri, Phys. Rev. Lett. **95**, 115501 (2005).
[4] M. Kachanov and I. Sevostianov, Int. J. Solids Struct. **42**, 309 (2005).
[5] *Heterogeneous Media: Micromechanics, Modeling, Methods, and Simulations*, edited by K.Z. Markov and L. Preziozi (Birkhauser, Boston, 2000).
[6] L.J. Walpole, Adv. Appl. Mech. **11**, 169 (1981).
[7] J.G. Berryman, J. Acoust. Soc. Am. **68**, 1820 (1980).
[8] M. Avellaneda, Commun. Pure Appl. Math. **40**, 527 (1987).
[9] R. McLaughlin, Int. J. Eng. Sci. **15**, 237 (1977).
[10] S. Giordano, Eur. J. Mech. A. Solids **22**, 885 (2003).
[11] R. Lakes, Science **235**, 1038 (1987).
[12] R. Lakes, Phys. Rev. Lett. **86**, 2897 (2001).
[13] T. Pointer, E. Liu, and J.A. Hudson, Geophys. J. Int. **142**, 199 (2000).
[14] S.R. Tod, Geophys. J. Int. **146**, 249 (2001).
[15] R.K. Nalla, J.H. Kinney, and R.O. Ritchie, Biomaterials **24**, 3955 (2003).
[16] N.A. Danova *et al.*, Bone (N.Y.) **33**, 197 (2003).
[17] M. Kachanov, Appl. Mech. Rev. **45**, 305 (1992).
[18] J.D. Eshelby, Proc. R. Soc. A **241**, 376 (1957).
[19] T. Mura, *Micromechanics of Defects in Solids* (Kluwer Academic Publishers, Dordrecht, 1991).
[20] M. Kachanov, Adv. Appl. Mech. **30**, 259 (1994).