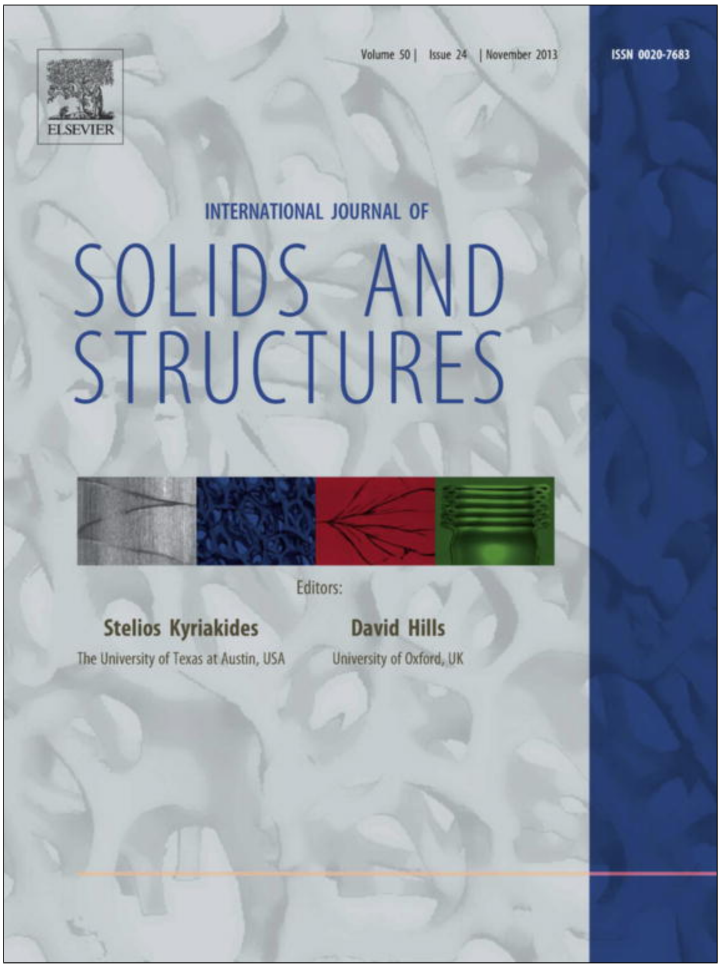
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Analytical procedure for determining the linear and nonlinear effective properties of the elastic composite cylinder

Stefano Giordano

Joint International Laboratory LICS/LEMAC, Institute of Electronics, Microelectronic and Nanotechnology (UMR CNRS 8520), PRES University North of France, ECLille, Avenue Poincaré, BP 60069, 59652 Villeneuve d'Ascq, France

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In this work we consider a cylindrical structure composed of a nonlinear core (inhomogeneity) surrounded by a different nonlinear shell (matrix). We elaborate a technique for determining its linear elastic moduli (second order elastic constants) and the nonlinear elastic moduli, which are called Landau coefficients (third order elastic constants). Firstly, we develop a nonlinear perturbation method which is able to turn the initial nonlinear elastic problem into a couple of linear problems. Then, we prove that only the solution of the first linear problem is necessary to calculate the linear and nonlinear effective properties of the heterogeneous structure. The following step consists in the exact solution of such a linear problem by means of the complex elastic potentials. As result we obtain the exact closed forms for the linear and nonlinear effective elastic moduli, which are valid for any volume fraction of the core embedded in the external shell.

1. Introduction

The linear effective properties of heterogeneous structures and composite materials have been extensively studied and many micromechanical models have been proposed for different microstructures (Nemat-Nasser and Hori, 1999; Torquato, 2002; Milton, 2002). The Eshelby results (Eshelby, 1957, 1959) and its generalizations have been found to be useful in the determination of the effective physical properties. In fact, the homogenization procedures contain at first the exact mathematical analysis of the mechanical behavior induced by a single inhomogeneity (Mura, 1987; Christensen, 2005; Asaro and Lubarda, 2006), and then proceed by considering the more general case of interacting particles (Hill, 1963; Hashin, 1983). This approach has been carried out in the limit of a low density defect population by Mori and Tanaka (1973). Such an hypothesis can be partially removed by means of different methods, such as iterated homogenizations and differential schemes (McLaughlin, 1977; Giordano, 2003). These techniques have been applied with great accuracy both to the case of embedded inhomogeneities by Snyder and Garboczi (1992) and Kachanov and Sevostianov (2005) and to the case of dispersed defects, such as micro-cracks in a matrix (Budiansky and O'Connell, 1976; Kachanov, 1992; Giordano and Colombo, 2007a,b).

Typically, these methods have been developed for determining the effective linear elastic properties starting from the linear elastic properties of the components. A methodology based on the Eshelby heritage has been proposed for analysing the effective nonlinear properties of dispersions of nonlinear particles embedded in a linear matrix (Giordano et al., 2008, 2009; Colombo and Giordano, 2011). More general results, based on variational principles for nonlinear materials, have been obtained by Talbot and Willis (1985a, 1987b). Variational methods for deriving improved bounds and estimates for nonlinear media, utilizing linear elastic comparison materials, were introduced by Ponte Castañeda (1991a, 1992b,c, 1996) and Suquet (1993). These methodologies can be found in a complete review by Ponte Castañeda and Suquet (1998). A variational formulation was also adopted in order to derive explicit nonlocal constitutive equations for a class of random composite materials (Drugan and Willis, 1996).

In this paper we take into consideration a specific structure with two different nonlinear phases: a cylinder composed of a nonlinear core (or inhomogeneity) embedded into a nonlinear shell (or matrix). The geometry of the system is depicted in Fig. 1. We introduce a homogenization technique for the linear elastic moduli (second order elastic constants) and for the nonlinar elastic moduli, which are called Landau coefficients (third order elastic constants). The proposed procedure is based on two main steps: firstly, we develop a nonlinear perturbation method which is able to turn the initial nonlinear elastic problem into a couple of linear problems, which are simpler and analytically solvable. Then, we are able to prove that only the solution of the first linear problem is needed for determining the linear and nonlinear effective properties of the heterogeneous structure. The second step consists in the exact solution of such a linear problem (within

E-mail address: Stefano.Giordano@iemn.univ-lille1.fr

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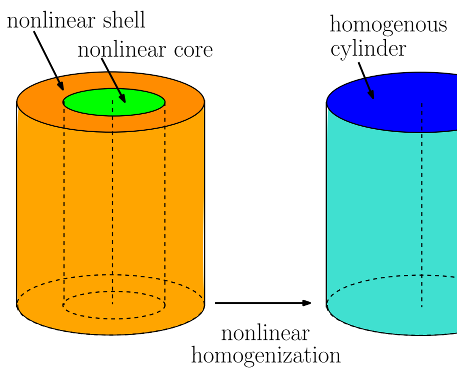


Fig. 1. Schematic representation of the composite cylinder consisting of a nonlinear core in a nonlinear shell: through a homogenization procedure it is possible to define an equivalent nonlinear homogeneous cylinder.

the two-dimensional elasticity) by means of the Kolossov–Muskhelishvili elastic potentials (Muskhelishvili, 1953). As final result we obtain the exact closed forms for the linear and nonlinear effective elastic moduli, which are valid for any volume fraction of the core embedded in the external shell.

Analytical results for coated fibers and composite cylinders are important for several applications in material science. From the linear elasticity point of view important results for the multi-shell cylinder were found by Stucu (1992). Moreover, the elastic behavior of multiply coated fibre-reinforced composites was investigated by Hervé and Zaoui (1995). Afterwards, this model has been generalized in order to consider the thermal and thermoelastic behavior of these structures (Hervé, 2002). A complete analysis of the effects of inhomogeneous inter-phases between fibers and matrix has been developed by Shen and Li (2003). Recent generalizations take into account composite cylinders with arbitrary anisotropic constitution (Shokrolahi-Zadeh and Shodja, 2008), with cylindrically orthotropic layers (Tsukrov and Drach, 2010) and fibrous composites of piezoelectric and piezomagnetic phases (Kuo, 2011).

In spite of the development of refined techniques for analysing the linear properties of composite cylinders and fibrous structures, many applications to real materials need to deal with nonlinear features and their mixing laws. For example, multi-shell nanowires are candidates for future electronic and photonic devices. Indeed, electronic confinement in two and three dimensions have been realized through quantum wires and quantum dots, respectively (Johnson and Freund, 2001; Lauhon et al., 2004). The quantitative knowledge of stress and strain distributions in these nonlinear structures are essential for characterizing and tailoring their optoelectronic properties (Johnson and Freund, 2001). A second example of great importance in material science concerns the use of single-walled and multi-walled nanotubes in reinforced composites, as described, e.g., by Seidel and Lagoudas (2006). The effective properties of these structures can be analysed through the composite cylinder approach, where the nonlinear features must be taken into consideration (Qian et al., 2002). Such methods can be also used for determining the vibrational behavior of nanotube-reinforced panels (Aragh et al., 2012) and for analysing the mechanical buckling of aligned nanotubes (Mehrabadi et al., 2012).

For an accurate evaluation of the physical properties of previous systems it is necessary to account for the elastic nonlinearity of their constituents and to understand the corresponding mixing rules. We underline that nonlinearity can be introduced in the theory of elasticity by means of the exact relation for the Lagrangian strain (geometrical nonlinearity) and/or through a nonlinear stress–strain constitutive relation (physical nonlinearity). In this work, we adopt the physical nonlinearity standpoint, whereas the geometrical nonlinearity is not considered (hypothesis of small deformations).

The structure of the paper is the following: in Section 2 we introduce the nonlinear perturbation method. In Section 3 we define the details of the system under investigation: the cylindrical composite system. Then, we analyse the linear problem in Section 3.1 and the nonlinear one in Section 3.2. Finally in Section 4 we show some applications and comparisons with earlier theories.

2. Nonlinear perturbation method

We consider a nonlinear elastic problem for an arbitrary region $\Omega$with boundary $\partial\Omega$ and unit normal vector $\vec{n}$ (see Fig. 2 for details). Under the hypothesis of small deformations we can introduce the displacement field $\vec{u}(\vec{x})$, the infinitesimal strain tensor $\hat{\epsilon}(\vec{x})$ and the Cauchy stress tensor $\hat{T}(\vec{x})$. The standard definition of the strain tensor follows

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

and the balance laws for the linear and angular momentum are given below (for the static case without body forces)

$$\frac{\partial T_{ji}}{\partial x_i} = 0, \quad T_{ij} = T_{ji} \tag{2}$$

These laws hold for all materials, regardless of their constitution. However, in order to obtain a complete system of equations we need to introduce the constitutive equations, which characterize the actual elastic behaviour of the investigated system. Here, we are interested in an heterogeneous nonlinear behaviour in Ω and, therefore, we assume the following constitutive equation

$$T_{ij}(\vec{x}) = C_{ijkh}(\vec{x})\epsilon_{kh}(\vec{x}) + \lambda N_{ijkhlm}(\vec{x})\epsilon_{kh}(\vec{x})\epsilon_{lm}(\vec{x}) \tag{3}$$

or, equivalently, in compact tensor notation $\hat{T}(\vec{x}) = \hat{C}(\vec{x})\hat{\epsilon}(\vec{x}) + \lambda\hat{N}(\vec{x})\hat{\epsilon}(\vec{x})\hat{\epsilon}(\vec{x})$ where $\hat{C}$ describes the linear elasticity (the C_{ijkh} are the second order elastic constants, SOEC) while $\hat{N}$ describes the nonlinear elasticity (the N_{ijkhlm} are the third order elastic constants, TOEC). Both $\hat{C}$ and $\hat{N}$ are heterogeneous tensors over Ω. The tensor $\hat{N}$ measures the first deviation from the linearity. It can be noticed that the tensor $\hat{C}$ has 21 independent entries, while the second order tensor $\hat{N}$ has 56 independent components. Tables for the values of C_{ijkh} and N_{ijkhlm} can be found in literature for different crystal symmetries (Ballabh et al., 1992). Moreover, these values can be obtained by experimental procedures (Hughes and Kelly, 1953) and by computational techniques, such as molecular dynamics (Cain and Ray, 1988) or first-principles calculations (Zhao et al., 2007). In order to apply a perturbation method we have introduced in Eq. (3) a small parameter λ. The general nonlinear problem in the

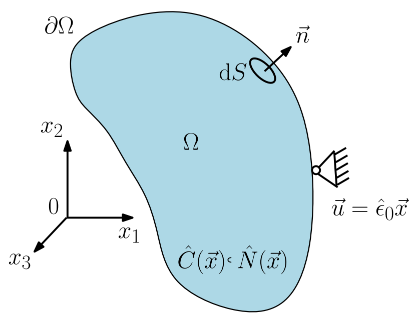


Fig. 2. Definition of a nonlinear elastic problem for a region Ω with prescribed displacement on the boundary $\partial\Omega$. The constitutive relation is controlled by $\hat{C}$, which describes the linear elasticity (tensor of the second order elastic constants, SOEC), while $\hat{N}$ represents the nonlinear response (tensor the third order elastic constants, TOEC). The unit vector $\vec{n}$ is normal to the external surface $\partial\Omega$ and it is associated with the area element dS.

region Ω is described by the combination of Eqs. (1)–(3) with the boundary condition $\vec{u} = \hat{\epsilon}_0\vec{x}$ on $\partial\Omega$ where $\hat{\epsilon}_0$ is a constant strain tensor (prescribed displacement on $\partial\Omega$). The formulated problem has a series of important properties which are thoroughly discussed in the following.

Firstly, we observe that if the system is homogeneous (i.e., $\hat{C}(\vec{x}) = \hat{C}$ and $\hat{N}(\vec{x}) = \hat{N}\,\forall\,\vec{x} \in \Omega$), then the solution is given by $\hat{\epsilon}(\vec{x}) = \hat{\epsilon}_0\,\forall\,\vec{x} \in \Omega$. In fact, the uniform strain $\hat{\epsilon}_0$ in Ωsatisfies the boundary conditions and leads to an uniform stress, compatible with the balance equations stated in Eq. (2).

As second property, we remember that for arbitrarily heterogeneous structures, the average value of the strain tensor over the region Ω is equal to $\hat{\epsilon}_0$, regardless of the constitutive equation. So, we have

$$\langle \epsilon_{ij} \rangle = \frac{1}{V}\int_\Omega \epsilon_{ij}(\vec{x})d\vec{x} = \epsilon_{0,ij} \tag{4}$$

where $V = \text{mes}(\Omega)$. This property is sometimes called average-strain theorem (Qu and Cherkaoui, 2006).

We can now define the effective linear and nonlinear elastic tensors of the heterogeneous structure through the determination of the average value of the stress tensor over the region Ω. Since $\langle\hat{\epsilon}\rangle = \hat{\epsilon}_0$ we can write

$$\left\langle \hat{T} \right\rangle = \hat{C}_{\text{eff}}\hat{\epsilon}_0 + \lambda\hat{N}_{\text{eff}}\hat{\epsilon}_0\hat{\epsilon}_0 + o(\lambda) \tag{5}$$

where we have neglected the terms with the third power of the strain and higher. Eq. (5) represents an operative definition of the effective response based on the evaluation of the average stress in the structure. This result comes from the fact that the stress is a homogeneous function of degree 1 and 2 of the applied average strain for the linear and nonlinear part, respectively.

The effective parameters $\hat{C}_{\text{eff}}$ and $\hat{N}_{\text{eff}}$ can be alternatively defined through the determination of the average value of the tensor contraction $\hat{T} : \hat{\epsilon} = T_{ij}\epsilon_{ij}$. In fact, they can be introduced through the following relation

$$\left\langle \hat{T} : \hat{\epsilon} \right\rangle = \hat{\epsilon}_0 : \hat{C}_{\text{eff}}\hat{\epsilon}_0 + \lambda\hat{\epsilon}_0 : \hat{N}_{\text{eff}}\hat{\epsilon}_0\hat{\epsilon}_0 + o(\lambda) \tag{6}$$

which represents, differently from Eq. (5), an operative definition of the effective response based on the evaluation of the average value of $\hat{T} : \hat{\epsilon}$ in the heterogeneous structure. The definitions of the effective behavior given in Eqs. (5) and (6) are mathematically equivalent. This point can be verified by using the following result

$$\langle T_{ij}\epsilon_{ij} \rangle = \left\langle \hat{T} \right\rangle : \hat{\epsilon}_0 \tag{7}$$

It represents the so-called Hill's lemma (Qu and Cherkaoui, 2006). It is evident that the substitution of Eq. (5) in Eq. (7) leads to Eq. (6), proving the equivalence of the different definitions of the effective properties.

In order to apply a perturbation technique, we consider λ as a small parameter and we search a solution of the general problem in the form

$$\vec{u} = \vec{u}^L + \lambda\vec{u}^{NL} + o(\lambda) \tag{8}$$
$$\hat{\epsilon} = \hat{\epsilon}^L + \lambda\hat{\epsilon}^{NL} + o(\lambda) \tag{9}$$
$$\hat{T} = \hat{T}^L + \lambda\hat{T}^{NL} + o(\lambda) \tag{10}$$

By using the constitutive equation (see Eq. (3)) we have

$$\hat{T} = \hat{C}\hat{\epsilon} + \lambda\hat{N}\hat{\epsilon}\hat{\epsilon} = \hat{C}\hat{\epsilon}^L + \lambda\left(\hat{C}\hat{\epsilon}^{NL} + \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L\right) + o(\lambda) \tag{11}$$

By applying the linear momentum balance equation we simply obtain

$$\vec{\nabla}\cdot\left(\hat{C}\hat{\epsilon}^L\right) + \lambda\vec{\nabla}\cdot\left(\hat{C}\hat{\epsilon}^{NL} + \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L\right) + o(\lambda) = 0 \quad \forall\lambda \tag{12}$$

Therefore, for the arbitrariness of λ, we obtain the standard linear problem of the elasticity theory

$$\begin{cases} \vec{\nabla}\cdot\left(\hat{C}\hat{\epsilon}^L\right) = 0 \\ \hat{\epsilon}^L = \frac{1}{2}\left(\vec{\nabla}\vec{u}^L + \vec{\nabla}^T\vec{u}^L\right) \\ \vec{u}^L = \hat{\epsilon}_0\vec{x} \quad \text{on } \partial\Omega \end{cases} \tag{13}$$

and the associated problem which describes the first deviation from the linearity

$$\begin{cases} \vec{\nabla}\cdot\left(\hat{C}\hat{\epsilon}^{NL}\right) = -\vec{\nabla}\cdot\left(\hat{N}\hat{\epsilon}^L\hat{\epsilon}^L\right) \\ \hat{\epsilon}^{NL} = \frac{1}{2}\left(\vec{\nabla}\vec{u}^{NL} + \vec{\nabla}^T\vec{u}^{NL}\right) \\ \vec{u}^{NL} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{14}$$

The initial nonlinear problem has been split in two simpler linear problems. The first one is the standard linear elasticity problem and the second one is a linear problem with a distribution of body forces, depending on the solution of the first problem. The solution of these problems allows us to describe the elastic fields within the heterogeneous structure. This method could be also generalized in order to consider more terms in the series expansion: it would lead to a hierarchy of Lamé equations describing the behaviour of the displacement expansion terms.

In the present work, we attempt an alternative approach useful to calculate directly the elastic effective properties of the composite body (at least the SOEC and the TOEC). We consider again the tensor contraction $\hat{T} : \hat{\epsilon}$ and we expand it in series of λ (by using Eqs. (8) and (11))

$$\left\langle \hat{T} : \hat{\epsilon} \right\rangle = \left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^L \right\rangle + \lambda\left\langle 2\hat{\epsilon}^L : \hat{C}\hat{\epsilon}^{NL} + \hat{\epsilon}^L : \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L \right\rangle + o(\lambda) \tag{15}$$

We can now prove that $\left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^{NL} \right\rangle = 0$ as follows. To begin we observe that

$$\left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^{NL} \right\rangle = \frac{1}{V}\int_\Omega C_{ijkl}\frac{\partial u_i^L}{\partial x_j}\frac{\partial u_k^{NL}}{\partial x_l}d\vec{x} \tag{16}$$

because of the symmetries of the linear elastic tensor. Therefore we have

$$\left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^{NL} \right\rangle = \frac{1}{V}\int_\Omega \frac{\partial}{\partial x_l}\left(C_{ijkl}\frac{\partial u_i^L}{\partial x_j}u_k^{NL}\right)d\vec{x}$$
$$-\frac{1}{V}\int_\Omega \frac{\partial}{\partial x_l}\left(C_{ijkl}\frac{\partial u_i^L}{\partial x_j}\right)u_k^{NL}d\vec{x} \tag{17}$$

The second integral is zero since $\partial T_{kl}^L/\partial x_l = 0$ and we obtain

$$\left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^{NL} \right\rangle = \frac{1}{V}\int_{\partial\Omega} C_{ijkl}\frac{\partial u_i^L}{\partial x_j}u_k^{NL}n_l dS = 0 \tag{18}$$

since $\vec{u}^{NL} = 0$ on $\partial\Omega$. To conclude, starting with Eq. (15) we have obtained the result

$$\left\langle \hat{T} : \hat{\epsilon} \right\rangle = \left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^L \right\rangle + \lambda\left\langle \hat{\epsilon}^L : \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L \right\rangle + o(\lambda) \tag{19}$$

Drawing a comparison with Eq. (6) we obtain the main achievements of the present section

$$\hat{\epsilon}_0 : \hat{C}_{\text{eff}}\hat{\epsilon}_0 = \left\langle \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^L \right\rangle \tag{20}$$
$$\hat{\epsilon}_0 : \hat{N}_{\text{eff}}\hat{\epsilon}_0\hat{\epsilon}_0 = \left\langle \hat{\epsilon}^L : \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L \right\rangle \tag{21}$$

In previous expressions $\hat{\epsilon}^L(\vec{x})$ is the solution of the linear problem stated in Eq. (13) with the boundary condition defined by the constant strain $\hat{\epsilon}_0$. Therefore, linear and nonlinear effective properties (i.e., SOEC and TOEC) depend only on the linear solution of the proposed problem. We remark that Eqs. (20) and (21) are exact results, not affected by any kind of approximation. In fact, the approximation introduced by the perturbation technique concerns the number of terms retained in the series expansions, being each term exactly evaluated. Moreover, we remark that the knowledge of the linear solution $\hat{\epsilon}^L$ is adequate only to obtain the second order and the third order effective elastic constants, but it is not sufficient for determining the higher order effective behaviors. Evidently, in such a case it is necessary to consider also the nonlinear solution $\hat{\epsilon}^{NL}$ and the further terms of the series expansion. To conclude, our methodology is based on the following idea: by selecting different suitable homogeneous deformations $\hat{\epsilon}_0$ and by solving the correspondent linear problem given in Eq. (13), we can efficiently apply Eqs. (20) and (21) and we can find all the components of the effective tensors $\hat{C}_{\rm eff}$ and $\hat{N}_{\rm eff}$. In the following sections we apply Eqs. (20) and (21) to an isotropic nonlinear composite cylinder under plane-strain conditions.

3. The cylindrical composite system

We are interested in applying the previous procedure to the case of a composite cylinder composed by a nonlinear core embedded in a nonlinear external shell (as in Fig. 1). We suppose to analyse the plane strain behavior of this heterogeneous structure on a plane perpendicular to the axis of the cylinder (see Fig. 3). To begin we define the linear and nonlinear elastic behavior of both components. We consider each phase ($\alpha = 1,2$) described by the Green approach (Atkin and Fox, 2005) and we formulate the corresponding energy balance (Landau and Lifschitz, 1959): for a given state of deformation, the stress power is absorbed into a strain energy function $U_\alpha(\hat{\epsilon})$, leading to the constitutive equation $\hat{T} = \partial U_\alpha(\hat{\epsilon})/\partial\hat{\epsilon}$. As well known, the strain energy function can be identified with the internal energy per unit volume in an isentropic process, or with the Helmholtz free-energy per unit volume in an isothermal process. To model core and shell materials, we adopt the most general isotropic nonlinear constitutive stress–strain relation of the two-dimensional elasticity, expanded up to the second order in the strain components: it follows that the function $U_\alpha(\hat{\epsilon})$ can only depend upon the principal invariants of the strain tensor, i.e., $U_\alpha(\hat{\epsilon}) = U_\alpha({\rm Tr}(\hat{\epsilon}), {\rm Tr}(\hat{\epsilon}^2))$.

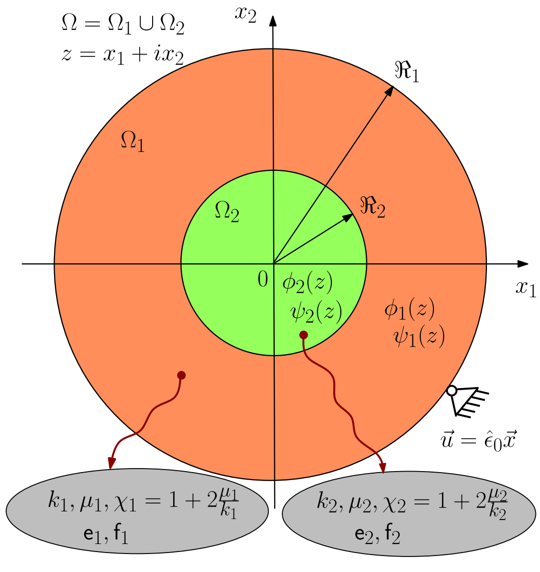


Fig. 3. Schematic representation of the cross-section of the composite cylinder shown in Fig. 1. The whole planar region Ω has been subdivided into the shell Ω_1 (with parameters k_1, μ_1, χ_1, e_1 and f_1) and the core Ω_2 (with parameters k_2, μ_2, χ_2, e_2 and f_2). We will adopt the complex variable method for the two-dimensional elasticity based on the couple of Kolossov–Muskhelishvili potentials $\phi_\alpha(z)$ and $\psi_\alpha(z)$ ($\alpha = 1,2$), where the complex number $z = x_1 + ix_2$ represents the position on the plane.

Therefore, by expanding $U_\alpha(\hat{\epsilon})$ up to the third order in the strain components, we obtain

$$U_\alpha(\hat{\epsilon}) = \mu_\alpha {\rm Tr}(\hat{\epsilon}^2) + \frac{1}{2}(k_\alpha - \mu_\alpha){\rm Tr}^2(\hat{\epsilon}) + e_\alpha {\rm Tr}(\hat{\epsilon}){\rm Tr}(\hat{\epsilon}^2) + f_\alpha {\rm Tr}^3(\hat{\epsilon}) \quad (22)$$

and deriving the stress, we get

$$\hat{T} = 2\mu_\alpha\hat{\epsilon} + (k_\alpha - \mu_\alpha){\rm Tr}(\hat{\epsilon})\hat{I} + 2e_\alpha {\rm Tr}(\hat{\epsilon})\hat{\epsilon} + e_\alpha {\rm Tr}(\hat{\epsilon}^2)\hat{I} + 3f_\alpha {\rm Tr}^2(\hat{\epsilon})\hat{I} \quad (23)$$

for the materials corresponding to the shell ($\alpha = 1$) and to the core ($\alpha = 2$). We remark that in Eqs. (22) and (23) tensors $\hat{\epsilon}$ and $\hat{T}$ are represented by matrices 2x2 (planar geometry). The parameters e_α and f_α are the so-called (two-dimensional) Landau moduli (Landau and Lifschitz, 1959; Colombo and Giordano, 2011) and they represent the first deviation from the standard linearity (they are the TOEC in our system). On the other hand, k_α and μ_α represent the linear elastic moduli or, equivalently, the SOEC. Parameters k_α represent the two-dimensional bulk moduli and therefore they are related with the standard three-dimensional bulk moduli K_α through the relations $k_\alpha = K_\alpha + (1/3)\mu_\alpha$ ($\alpha = 1,2$). On the other hand, the shear moduli μ_α assume the same value both in the two-dimensional and three-dimensional geometries and, therefore, there is no possible ambiguity. Of course, we assume that the technological assembling processes are able to generate a quite perfect core–shell interface. In fact, it is well known that the behavior of composite materials (in particular at the nanoscale) is deeply affected by interface features occurring at the boundary between different phases (Palla et al., 2008, 2009, 2010; Giordano et al., 2012). In the following sections we search for the overall behavior of the cylinder, which is described by the effective constitutive equation deduced from Eq. (5)

$$\langle\hat{T}\rangle = 2\mu_{\rm eff}\langle\hat{\epsilon}\rangle + (k_{\rm eff} - \mu_{\rm eff}){\rm Tr}(\langle\hat{\epsilon}\rangle)\hat{I} + 2e_{\rm eff}{\rm Tr}(\langle\hat{\epsilon}\rangle)\langle\hat{\epsilon}\rangle + e_{\rm eff}{\rm Tr}\left(\langle\hat{\epsilon}\rangle^2\right)\hat{I} + 3f_{\rm eff}{\rm Tr}^2(\langle\hat{\epsilon}\rangle)\hat{I} + o\left(\|\langle\hat{\epsilon}\rangle\|^2\right) \quad (24)$$

where we have defined the effective linear moduli $k_{\rm eff}$ and $\mu_{\rm eff}$ and the effective nonlinear coefficients $e_{\rm eff}$ and $f_{\rm eff}$. The effective constitutive relation given in Eq. (24) has the same form of Eq. (23) which is valid for each phase, except for the possible higher order terms, which will be neglected in the following developments. All the linear ($k_{\rm eff}$ and $\mu_{\rm eff}$) and nonlinear ($e_{\rm eff}$ and $f_{\rm eff}$) effective elastic parameters will be found as function of the parameters of the two phases and the volume fraction $c = \Re_2^2/\Re_1^2$ of the core (with radius $\Re_2$) embedded in the shell (with external radius $\Re_1$).

3.1. Linear analysis

In order to solve the linear counterpart of our problem, we use the complex variable method for the two-dimensional elasticity (Atkin and Fox, 2005; Muskhelishvili, 1953). We assume that the elastic state of a given homogeneous region Ω_α ($\alpha = 1,2$) is exactly described by two holomorphic functions $\phi_\alpha(z)$ and $\psi_\alpha(z)$, where $z$represents the position on the plane (see Fig. 3 for details). The Kolossov–Muskhelishvili equations allow for the determination of the elastic fields in each region

$$u_1^\alpha + i\,u_2^\alpha = \frac{1}{2\mu_\alpha}\left[\chi_\alpha\phi_\alpha(z) - z\overline{\phi'_\alpha(z)} - \overline{\psi_\alpha(z)}\right] \quad (25)$$

$$T_{11}^\alpha + T_{22}^\alpha = 2\left[\phi'_\alpha(z) + \overline{\phi'_\alpha(z)}\right] \quad (26)$$

$$T_{22}^\alpha - T_{11}^\alpha + 2\,i\,T_{12}^{(\alpha)} = 2\left[\bar{z}\phi''_\alpha(z) + \psi'_\alpha(z)\right] \quad (27)$$

where $\bar{f}$ is the conjugate of f while f' and f'' indicate the first and the second derivative of the analytic function f, respectively. Functions $\phi_1(z)$ and $\psi_1(z)$ are defined for $\Re_2 <| z |< \Re_1$, while $\phi_2(z)$ and $\psi_2(z)$

are defined for $|z| < \Re_2$. Moreover, the parameter χ_α introduced in Eq. (27) is given by $\chi_\alpha = 3 - 4\nu_\alpha$ (plane strain conditions) where the Poisson ratio can be written as $\nu_\alpha = (k_\alpha - \mu_\alpha)/(2k_\alpha)$.

The solution of the elastic problem can be obtained by imposing the perfect bonding at the interface and the prescribed displacement on the external boundary. These conditions can be expressed in terms of the elastic potentials

$$\frac{1}{2\mu_1}\left[\chi_1\phi_1 - z\overline{\phi_1'} - \overline{\psi_1}\right] = \frac{1}{2\mu_2}\left[\chi_2\phi_2 - z\overline{\phi_2'} - \overline{\psi_2}\right] \quad (28)$$

$$\phi_1 + z\overline{\phi_1'} + \overline{\psi_1} = \phi_2 + z\overline{\phi_2'} + \overline{\psi_2} \quad (29)$$

for $z = \Re_2 e^{i\theta}$, and

$$\frac{1}{2\mu_1}\left[\chi_1\phi_1 - z\overline{\phi_1'} - \overline{\psi_1}\right] = \sum_{k=-\infty}^{+\infty} g_k e^{ik\theta} \quad (30)$$

for $z = \Re_1 e^{i\theta}$. Here, we have developed the prescribed displacement in Fourier series with coefficients g_k.

The potentials $\phi_2(z)$ and $\psi_2(z)$ can be represented by Taylor series and $\phi_1(z)$ and $\psi_1(z)$ by Laurent series (Atkin and Fox, 2005)

$$\phi_2(z) = \sum_{k=0}^{+\infty} a_k z^k, \quad \psi_2(z) = \sum_{k=0}^{+\infty} b_k z^k \quad (31)$$

$$\phi_1(z) = \sum_{k=-\infty}^{+\infty} c_k z^k, \quad \psi_1(z) = \sum_{k=-\infty}^{+\infty} d_k z^k \quad (32)$$

By substituting the series expansions in Eq. (28) and by separating the terms of the same power, we obtain

$$-\infty < k \leqslant -1 \Rightarrow \begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_k \Re_2^k - (2-k)\bar{c}_{2-k}\Re_2^{2-k} - \bar{d}_{-k}\Re_2^{-k}\right] \\ \quad = \frac{1}{2\mu_2}\left[-(2-k)\bar{a}_{2-k}\Re_2^{2-k} - \bar{b}_{-k}\Re_2^{-k}\right] \end{cases} \quad (33)$$

$$k = 0 \Rightarrow \begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_0 - 2\bar{c}_2\Re_2^2 - \bar{d}_0\right] \\ \quad = \frac{1}{2\mu_2}\left[\chi_2 a_0 - 2\bar{a}_2\Re_2^2 - \bar{b}_0\right] \end{cases} \quad (34)$$

$$k = 1 \Rightarrow \begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_1\Re_2 - \bar{c}_1\Re_2 - \bar{d}_{-1}\Re_2^{-1}\right] \\ \quad = \frac{1}{2\mu_2}\left[\chi_2 a_1\Re_2 - \bar{a}_1\Re_2\right] \end{cases} \quad (35)$$

$$2 \leqslant k < +\infty \Rightarrow \begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_k \Re_2^k - (2-k)\bar{c}_{2-k}\Re_2^{2-k} - \bar{d}_{-k}\Re_2^{-k}\right] \\ \quad = \frac{1}{2\mu_2}\left[\chi_2 a_k \Re_2^k\right]. \end{cases} \quad (36)$$

The same process can be applied to Eq. (29), by getting

$$-\infty < k \leqslant -1 \Rightarrow \begin{cases} c_k\Re_2^k + (2-k)\bar{c}_{2-k}\Re_2^{2-k} + \bar{d}_{-k}\Re_2^{-k} \\ \quad = (2-k)\bar{a}_{2-k}\Re_2^{2-k} + \bar{b}_{-k}\Re_2^{-k} \end{cases} \quad (37)$$

$$k = 0 \Rightarrow \begin{cases} c_0 + 2\bar{c}_2\Re_2^2 + \bar{d}_0 \\ \quad = a_0 + 2\bar{a}_2\Re_2^2 + \bar{b}_0 \end{cases} \quad (38)$$

$$k = 1 \Rightarrow \begin{cases} c_1\Re_2 + \bar{c}_1\Re_2 + \bar{d}_{-1}\Re_2^{-1} \\ \quad = a_1\Re_2 + \bar{a}_1\Re_2 \end{cases} \quad (39)$$

$$2 \leqslant k < +\infty \Rightarrow \begin{cases} c_k\Re_2^k + (2-k)\bar{c}_{2-k}\Re_2^{2-k} + \bar{d}_{-k}\Re_2^{-k} \\ \quad = a_k\Re_2^k \end{cases} \quad (40)$$

and, finally to Eq. (30), by obtaining

$$-\infty < k < +\infty \Rightarrow \frac{1}{2\mu_1}\left[\chi_1 c_k\Re_1^k - (2-k)\bar{c}_{2-k}\Re_1^{2-k} - \bar{d}_{-k}\Re_1^{-k}\right] = g_k \quad (41)$$

Now we determine a property which is valid in the range $2 \leqslant k < +\infty$; we substitute Eq. (40) in Eq. (36) and we use Eq. (41) in the result. We eventually obtain

$$c_k\left[\Re_2^k\left(\frac{\chi_1}{\mu_1} - \frac{\chi_2}{\mu_2}\right) - \frac{\Re_1^{2k}}{\Re_2^k}\chi_1\left(\frac{1}{\mu_1} + \frac{\chi_2}{\mu_2}\right)\right] - (2-k)\bar{c}_{2-k}\frac{\Re_2^2 - \Re_1^2}{\Re_2^k}\left(\frac{1}{\mu_1} + \frac{\chi_2}{\mu_2}\right) + 2\frac{\Re_1^k}{\Re_2^k}\left(\frac{1}{\mu_1} + \frac{\chi_2}{\mu_2}\right)\mu_1 g_k = 0 \quad (42)$$

Similarly, we can get a second property which is valid in the range $-\infty < k \leqslant -1$; we substitute Eq. (37) in Eq. (33) and we use Eq. (41) in the result. After a long but straightforward calculation we have

$$c_k\left[\Re_2^k\left(\frac{\chi_1}{\mu_1} + \frac{1}{\mu_2}\right) - \frac{\Re_1^{2k}}{\Re_2^k}\chi_1\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\right] - (2-k)\bar{c}_{2-k}\frac{\Re_2^2 - \Re_1^2}{\Re_2^k}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) + 2\frac{\Re_1^k}{\Re_2^k}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\mu_1 g_k = 0 \quad (43)$$

The kth relation in Eq. (42) or (43) depends only on the coefficients c_k, c_{2-k} and g_k. Therefore, it is evident that Eq. (42) for $k = 3$ and Eq. (43) for $k = -1$ form a closed system for the two unknowns c_{-1} and c_3; in addition, Eq. (42) for $k = 4$ and Eq. (43) for $k = -2$ form a closed system for the two unknowns c_{-2} and c_4, and so on. This procedure determines all the couples of unknowns (c_k, c_{2-k}) for $k \geqslant 3$. To complete the evaluation of the c_k series we have to determine c_2, c_1 and c_0. The coefficient c_2 (together with a_2 and d_{-2}) can be obtained through the system composed by Eqs. (36), (40) and (41) written for $k = 2$

$$\begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_2\Re_2^2 - \bar{d}_{-2}\Re_2^{-2}\right] = \frac{1}{2\mu_2}\chi_2 a_2\Re_2^2 \\ \frac{1}{2\mu_1}\left[\chi_1 c_2\Re_1^2 - \bar{d}_{-2}\Re_1^{-2}\right] = g_2 \\ c_2\Re_2^2 + \bar{d}_{-2}\Re_2^{-2} = a_2\Re_2^2 \end{cases} \quad (44)$$

We observe that the three unknowns of the previous systems are all zero if $g_2 = 0$ (as is in our case, see below). The coefficient c_1 (together with a_1 and d_{-1}) can be obtained through the system composed by Eqs. (35), (39) and (41) written for $k = 1$

$$\begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_1\Re_2 - \bar{c}_1\Re_2 - \bar{d}_{-1}\Re_2^{-1}\right] = \frac{1}{2\mu_2}\left[\chi_2 a_1\Re_2 - \bar{a}_1\Re_2\right] \\ c_1\Re_2 + \bar{c}_1\Re_2 + \bar{d}_{-1}\Re_2^{-1} = a_1\Re_2 + \bar{a}_1\Re_2 \\ \bar{d}_{-1} = \chi_1 c_1\Re_1^2 - \bar{c}_1\Re_1^2 - 2\mu_1\Re_1 g_1. \end{cases} \quad (45)$$

The coefficients c_0, a_0, b_0 and d_0 are governed by Eqs. (34), (38) and (41) written for $k = 0$

$$\begin{cases} \frac{1}{2\mu_1}\left[\chi_1 c_0 - 2\bar{c}_2\Re_2^2 - \bar{d}_0\right] = \frac{1}{2\mu_2}\left[\chi_2 a_0 - 2\bar{a}_2\Re_2^2 - \bar{b}_0\right] \\ c_0 + 2\bar{c}_2\Re_2^2 + \bar{d}_0 = a_0 + 2\bar{a}_2\Re_2^2 + \bar{b}_0 \\ \bar{d}_0 = \chi_1 c_0 - 2\bar{c}_2\Re_1^2 - 2\mu_1 g_0. \end{cases} \quad (46)$$

They can be all imposed to zero since $a_2 = 0$ and $c_2 = 0$ when $g_2 = 0$. Finally, the remaining unknowns of the sequences a_k, b_k and d_k can be directly found through Eqs. (40), (37) and (41), respectively. We have obtained the general solution for the elastic fields when the arbitrary applied displacement is given on the external boundary of the composite system.

Now, we impose $\vec{u} = \hat{\epsilon}_0\vec{x}$ on the external boundary. Hence, the sequence g_k is therefore characterized by $g_1 = \Re_1(\epsilon_{0,11} + \epsilon_{0,22})/2$ and $g_{-1} = \Re_1(\epsilon_{0,11} + 2i\epsilon_{0,12} - \epsilon_{0,22})/2$ and the remaining g_k are all zero. The previous analysis of the problem proves that the following simplified representations are sufficient to solve the problem

$$\psi_1(z) = d_{-3}\frac{1}{z^3} + d_{-1}\frac{1}{z} + d_1 z \quad (47)$$

$$\phi_1(z) = c_{-1}\frac{1}{z} + c_1 z + c_3 z^3 \quad (48)$$

$$\psi_2(z) = b_1 z \quad (49)$$

$$\phi_2(z) = a_1 z + a_3 z^3 \quad (50)$$

First of all we solve the system given in Eq. (45) for obtaining the coefficients c_1, a_1 and d_{-1}

$$a_1 = \frac{(\chi_1+1)\mathrm{Tr}(\hat{\epsilon}_0)}{\frac{1}{\mu_2}(2c+\chi_1-1)(\chi_2-1)+\frac{1}{\mu_1}2(1-c)(\chi_1-1)} \quad (51)$$

$$c_1 = \frac{\left[2+\frac{\mu_1}{\mu_2}(\chi_2-1)\right]\mathrm{Tr}(\hat{\epsilon}_0)}{\frac{1}{\mu_2}(2c+\chi_1-1)(\chi_2-1)+\frac{1}{\mu_1}2(1-c)(\chi_1-1)} \quad (52)$$

$$d_{-1} = \frac{\left[(\chi_1-1)-\frac{\mu_1}{\mu_2}(\chi_2-1)\right]2c\Re_1^2\mathrm{Tr}(\hat{\epsilon}_0)}{\frac{1}{\mu_2}(2c+\chi_1-1)(\chi_2-1)+\frac{1}{\mu_1}2(1-c)(\chi_1-1)} \quad (53)$$

As above said, Eq. (42) for $k=3$ and Eq. (43) for $k=-1$ form a closed system for the two unknowns c_{-1} and c_3: the solutions are

$$c_{-1} = \frac{-\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\mu_1(\epsilon_{0,11}+2i\epsilon_{0,12}-\epsilon_{0,22})\Re_2^2}{\frac{3}{\alpha}\left(\frac{1}{\mu_1}+\frac{\chi_2}{\mu_2}\right)\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\left(1-\frac{1}{c}\right)^2+\beta} \quad (54)$$

$$c_3 = -\bar{c}_{-1}\frac{1}{\alpha\Re_2^4}\left(\frac{1}{\mu_1}+\frac{\chi_2}{\mu_2}\right)\left(1-\frac{1}{c}\right) \quad (55)$$

where

$$\alpha = \left(\frac{\chi_1}{\mu_1}-\frac{\chi_2}{\mu_2}\right)-\frac{1}{c^3}\chi_1\left(\frac{1}{\mu_1}+\frac{\chi_2}{\mu_2}\right) \quad (56)$$

$$\beta = \left(\frac{\chi_1}{\mu_1}+\frac{1}{\mu_2}\right)-c\chi_1\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right) \quad (57)$$

The other coefficients can be eventually found as

$$d_{-3} = \chi_1\bar{c}_3\Re_1^6 + c_{-1}\Re_1^2 \quad (58)$$

$$d_1 = \chi_1\bar{c}_{-1}\Re_1^{-2} - 3c_3\Re_1^2 - \mu_1(\epsilon_{0,11}-2i\epsilon_{0,12}-\epsilon_{0,22}) \quad (59)$$

$$a_3 = c_3\left(1+\chi_1\frac{1}{c^3}\right) - \bar{c}_{-1}\frac{1}{\Re_2^4}\left(1-\frac{1}{c}\right) \quad (60)$$

$$b_1 = \bar{c}_{-1}\left[\frac{1}{\Re_2^2}\left(4-\frac{3}{c}\right)+\frac{\chi_1}{\Re_1^2}\right] - 3c_3\left(\Re_1^2+\frac{\chi_1\Re_2^2}{c^3}\right) - \mu_1(\epsilon_{0,11}-2i\epsilon_{0,12}-\epsilon_{0,22}) \quad (61)$$

We have solved an Eshelby-like problem in a finite region (the circle of radius $\Re_1$) and therefore it is possible to prove the perfect equivalence between our development and the two-dimensional Dirichlet–Eshelby tensor formalism introduced by Li et al. (2005) and Li and Wang (2008). The method of the complex potentials could be also used to obtain the two-dimensional Neumann-Eshelby tensor, useful to describe a set of prescribed forces on a finite region as discussed by Wang et al. (2005). We omit here this second approach for sake of brevity. However, the proposed solution is sufficient to apply the nonlinear homogenization technique. We remark that this technique can be also applied to the composite sphere model by using the three-dimensional version of the Eshelby tensor for finite regions, both under Dirichlet and Neumann boundary conditions (Li et al., 2007a,b). The final relations Eqs. (51)–(61) with $\Re_1 \to \infty$ are in perfect agreement with the classical two-dimensional Eshelby results (Hardiman, 1954; Eshelby, 1957, 1959). We observe that in the case of a finite region (i.e., finite radius $\Re_1$) the elastic strain is not uniform within the inhomogeneity (the coefficient $a_3 \neq 0$ generates a space-varying deformation), contrarily to the standard Eshelby theory ($\Re_1 \to \infty$) where the elastic fields are uniform within the embedded particle.

3.2. Nonlinear analysis

At this point we want to apply the nonlinear perturbation technique described in Section 2 to the composite cylinder system. To this aim we take into consideration the following linear isotropic operator identified through Eqs. (3) and (23) and described by the space varying linear moduli $\mu(\vec{x})$ and $k(\vec{x})$

$$\hat{C}(\vec{x})\hat{\epsilon}(\vec{x}) = 2\mu(\vec{x})\hat{\epsilon}(\vec{x}) + [k(\vec{x})-\mu(\vec{x})]\mathrm{Tr}[\hat{\epsilon}(\vec{x})]\hat{I} \quad (62)$$

Similarly, we consider the following nonlinear isotropic operator controlled by the space varying nonlinear coefficients $\mathsf{e}(\vec{x})$ and $\mathsf{f}(\vec{x})$

$$\hat{N}(\vec{x})\hat{\epsilon}(\vec{x})\hat{\epsilon}(\vec{x}) = 2\mathsf{e}(\vec{x})\mathrm{Tr}[\hat{\epsilon}(\vec{x})]\hat{\epsilon}(\vec{x}) + \mathsf{e}(\vec{x})\mathrm{Tr}\left\{[\hat{\epsilon}(\vec{x})]^2\right\}\hat{I} + 3\mathsf{f}(\vec{x})\mathrm{Tr}^2[\hat{\epsilon}(\vec{x})]\hat{I} \quad (63)$$

In order to use Eqs. (20) and (21) we determine the following scalar quantities for an arbitrary strain tensor field $\hat{\epsilon}(\vec{x})$

$$\hat{\epsilon}(\vec{x}):\hat{C}(\vec{x})\hat{\epsilon}(\vec{x}) = [k(\vec{x})+\mu(\vec{x})]\mathrm{Tr}^2[\hat{\epsilon}(\vec{x})] - 4\mu(\vec{x})\mathrm{Det}[\hat{\epsilon}(\vec{x})] \quad (64)$$

$$\hat{\epsilon}(\vec{x}):\hat{N}(\vec{x})\hat{\epsilon}(\vec{x})\hat{\epsilon}(\vec{x}) = 3[\mathsf{e}(\vec{x})+\mathsf{f}(\vec{x})]\mathrm{Tr}^3[\hat{\epsilon}(\vec{x})] - 6\mathsf{e}(\vec{x})\mathrm{Tr}[\hat{\epsilon}(\vec{x})]\mathrm{Det}[\hat{\epsilon}(\vec{x})] \quad (65)$$

In Eqs. (64) and (65) we used the standard relation $\mathrm{Tr}\left\{[\hat{\epsilon}(\vec{x})]^2\right\} = \mathrm{Tr}^2[\hat{\epsilon}(\vec{x})] - 2\mathrm{Det}[\hat{\epsilon}(\vec{x})]$ for two-dimensional matrices. Eqs. (64) and (65) must be evaluated with $\hat{\epsilon}(\vec{x}) = \hat{\epsilon}_0$, $\hat{C}(\vec{x}) = \hat{C}_{\mathrm{eff}}$ and $\hat{N}(\vec{x}) = \hat{N}_{\mathrm{eff}}$ for obtaining the left hand sides of Eqs. (20) and (21), and with $\hat{\epsilon}(\vec{x}) = \hat{\epsilon}^L$, $\hat{C}(\vec{x}) = \left\{\hat{C}_1 \text{ if } \vec{x}\in\Omega_1, \hat{C}_2 \text{ if } \vec{x}\in\Omega_2\right\}$ and $\hat{N}(\vec{x}) = \left\{\hat{N}_1 \text{ if } \vec{x}\in\Omega_1, \hat{N}_2 \text{ if } \vec{x}\in\Omega_2\right\}$ for obtaining the right hand sides of the same equations. The distribution of the linear part of the strain ($\hat{\epsilon}^L$) within our heterogeneous structure is easily determined with the results of Section 3.1: in fact, it is sufficient to consider the solutions Eqs. (47)–(50) and to substitute them in the displacement expression given in Eq. (25). By differentiating this last equation (see Eq. (1)), we easily find the strain tensor $\hat{\epsilon}^L$ within the regions Ω_1 and Ω_2, which is the main quantity exploited to obtain the linear and nonlinear effective elastic properties. It is important to remark that the $\hat{\epsilon}^L$ depends only on the strain $\hat{\epsilon}_0$ applied on the boundary of radius $\Re_1$ (prescribed displacement). For simplifying the formalism, we therefore introduce the scalar quantities $\mathsf{S}_C = \int_\Omega \hat{\epsilon}^L : \hat{C}\hat{\epsilon}^L d\vec{x}$ and $\mathsf{S}_N = \int_\Omega \hat{\epsilon}^L : \hat{N}\hat{\epsilon}^L\hat{\epsilon}^L d\vec{x}$, which are functions of the applied strain

$$\mathsf{S}_C(\hat{\epsilon}_0) = (k_1+\mu_1)\int_{\Omega_1}\left\{\mathrm{Tr}[\hat{\epsilon}^L(\vec{x})]\right\}^2 d\vec{x} - 4\mu_1\int_{\Omega_1}\mathrm{Det}[\hat{\epsilon}^L(\vec{x})]d\vec{x} + (k_2+\mu_2)\int_{\Omega_2}\left\{\mathrm{Tr}[\hat{\epsilon}^L(\vec{x})]\right\}^2 d\vec{x} - 4\mu_2\int_{\Omega_2}\mathrm{Det}[\hat{\epsilon}^L(\vec{x})]d\vec{x} \quad (66)$$

$$\mathsf{S}_N(\hat{\epsilon}_0) = 3(\mathsf{e}_1+\mathsf{f}_1)\int_{\Omega_1}\mathrm{Tr}^3[\hat{\epsilon}^L(\vec{x})]d\vec{x} - 6\mathsf{e}_1\int_{\Omega_1}\mathrm{Tr}[\hat{\epsilon}^L(\vec{x})]\mathrm{Det}[\hat{\epsilon}^L(\vec{x})]d\vec{x} + 3(\mathsf{e}_2+\mathsf{f}_2)\int_{\Omega_2}\mathrm{Tr}^3[\hat{\epsilon}^L(\vec{x})]d\vec{x} - 6\mathsf{e}_2\int_{\Omega_2}\mathrm{Tr}[\hat{\epsilon}^L(\vec{x})]\mathrm{Det}[\hat{\epsilon}^L(\vec{x})]d\vec{x} \quad (67)$$

All integrals in previous expressions can be calculated in closed form through the straightforward introduction of the cylindrical coordinates. However, we do not report here the complete results which are rather complicated and do not add relevant information to our development. Instead, we apply Eqs. (64) and (65) for determining the effective behavior.

We adopt suitable homogeneous in-plane deformations and we obtain the effective linear moduli

$$\mu_{\mathrm{eff}} = \frac{1}{4\pi\Re_1^2}\mathsf{S}_C\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) \quad (68)$$

$$k_{\mathrm{eff}} = \frac{1}{\pi\Re_1^2}\left\{\mathsf{S}_C\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) - \frac{1}{4}\mathsf{S}_C\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)\right\} \quad (69)$$

and the effective nonlinear coefficients

$$\mathsf{e}_{\text{eff}} = \frac{1}{3\pi\Re_1^2}\left\{2\mathsf{S}_N\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right) - \frac{1}{4}\mathsf{S}_N\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right\} \quad (70)$$

$$\mathsf{f}_{\text{eff}} = \frac{1}{3\pi\Re_1^2}\left\{\frac{1}{4}\mathsf{S}_N\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right) - \mathsf{S}_C\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right\} \quad (71)$$

Since the expression for $\mathsf{S}_C(\hat{\epsilon}_0)$ depends only on the linear elastic parameters, we observe that the effective linear moduli depend only on the linear elastic moduli of the components, as expected. On the other hand, it is evident that the effective nonlinear moduli depend both on the linear and nonlinear responses of the different phases composing the structure. It is interesting to observe that the whole procedure can be implemented both numerically, with standard software techniques, and analytically in symbolic computation environments. An explicit code has been developed with the Maple™ software and it can be found in the Additional Material Section. Here, all integrations defined above are calculated in

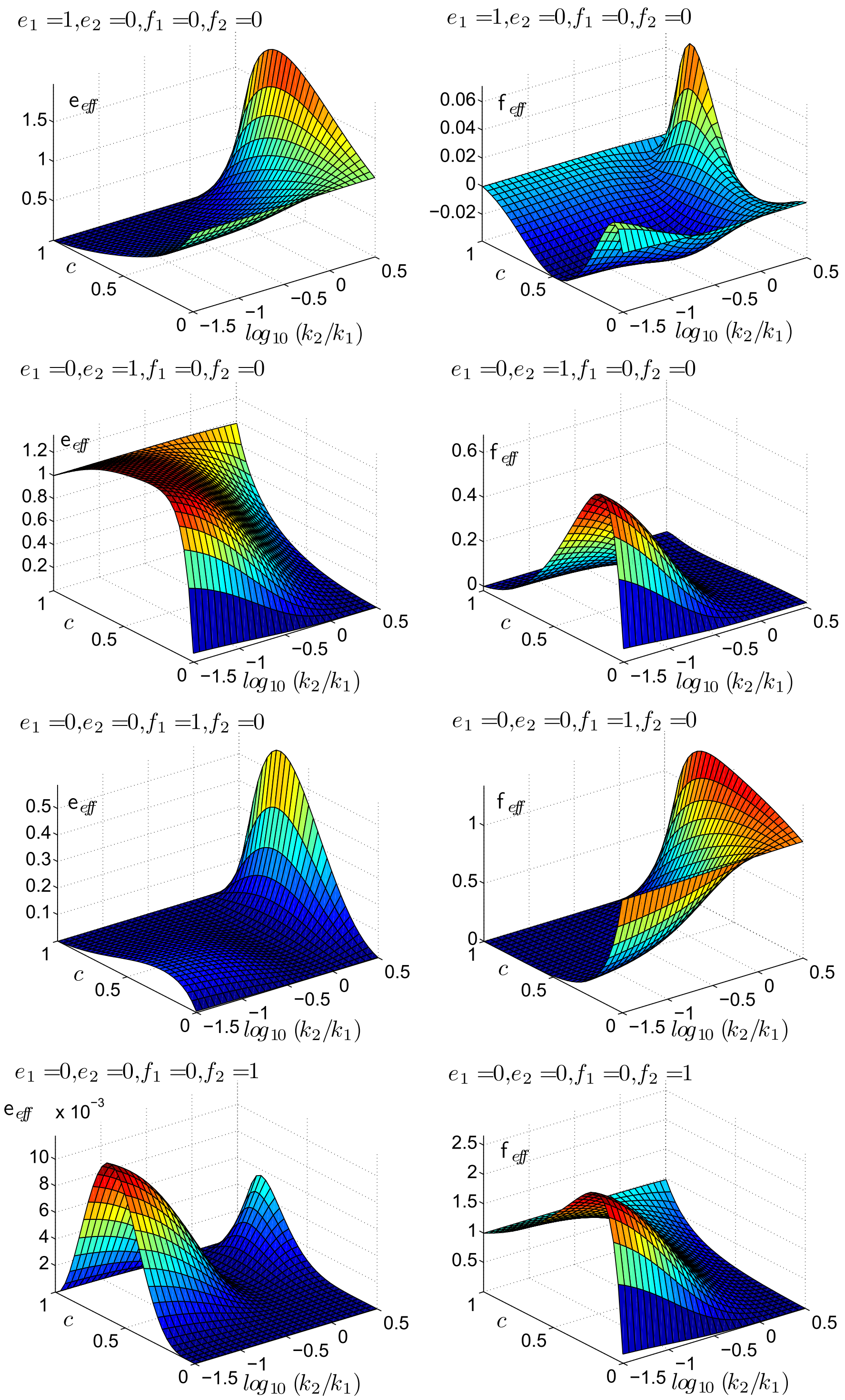


Fig. 4. Nonlinear effective elastic moduli in terms of the volume fraction c of the internal core and the compressibility contrast $\log_{10}(k_2/k_1)$. We have adopted the parameters $k_1 = 1$, $\nu_1 = 0.33$, $\nu_2 = 0.3$. Four couples of plots for e_{eff} and f_{eff} correspond to the nonlinear parameter of the structure ($\mathsf{e}_1 = 1$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 0$), ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 1$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 0$), ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 1$, $\mathsf{f}_2 = 0$) and ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 1$). It is therefore possible to observe the effects of any nonlinearity source separately.

closed form and all corresponding expressions are reported explicitly.

We show now a numerical application of Eqs. (68)–(71). We are interested in better understanding the effects of the nonlinear parameters e_1, f_1, e_2 and f_2 on the effective nonlinear properties of the overall composite cylinder. In Fig. 4 we can observe the behavior of e_{eff} and f_{eff} in terms of the volume fraction c of the internal core and the compressibility contrast $\log_{10}(k_2/k_1)$. We have adopted the parameters $k_1 = 1$, $\nu_1 = 0.33$, $\nu_2 = 0.3$. Four couples of plots for e_{eff} and f_{eff} correspond to the following nonlinear parameter of the structure: ($e_1 = 1$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$), ($e_1 = 0$, $e_2 = 1$, $f_1 = 0$, $f_2 = 0$), ($e_1 = 0$, $e_2 = 0$, $f_1 = 1$, $f_2 = 0$) and ($e_1 = 0$, $e_2 = 0$, $f_1 = 0$, $f_2 = 1$). It means that in each case we have considered only one source of nonlinearity in order to isolate its effects, produced on the effective nonlinear behavior. We can notice a very complex scenario from which we can deduce some general trends: (i) each of the nonlinearity e_1, f_1, e_2 and f_2 generates both e_{eff} and f_{eff} as final effective result; (ii) when the nonlinear behavior is concentrated in the matrix we observe a strong intensification of the effective nonlinearities for a positive contrast $\log_{10}(k_2/k_1)$ (i.e., $k_2 \gg k_1$); (iii) conversely, when the nonlinear behavior is concentrated in the core we observe a strong intensification of the effec-

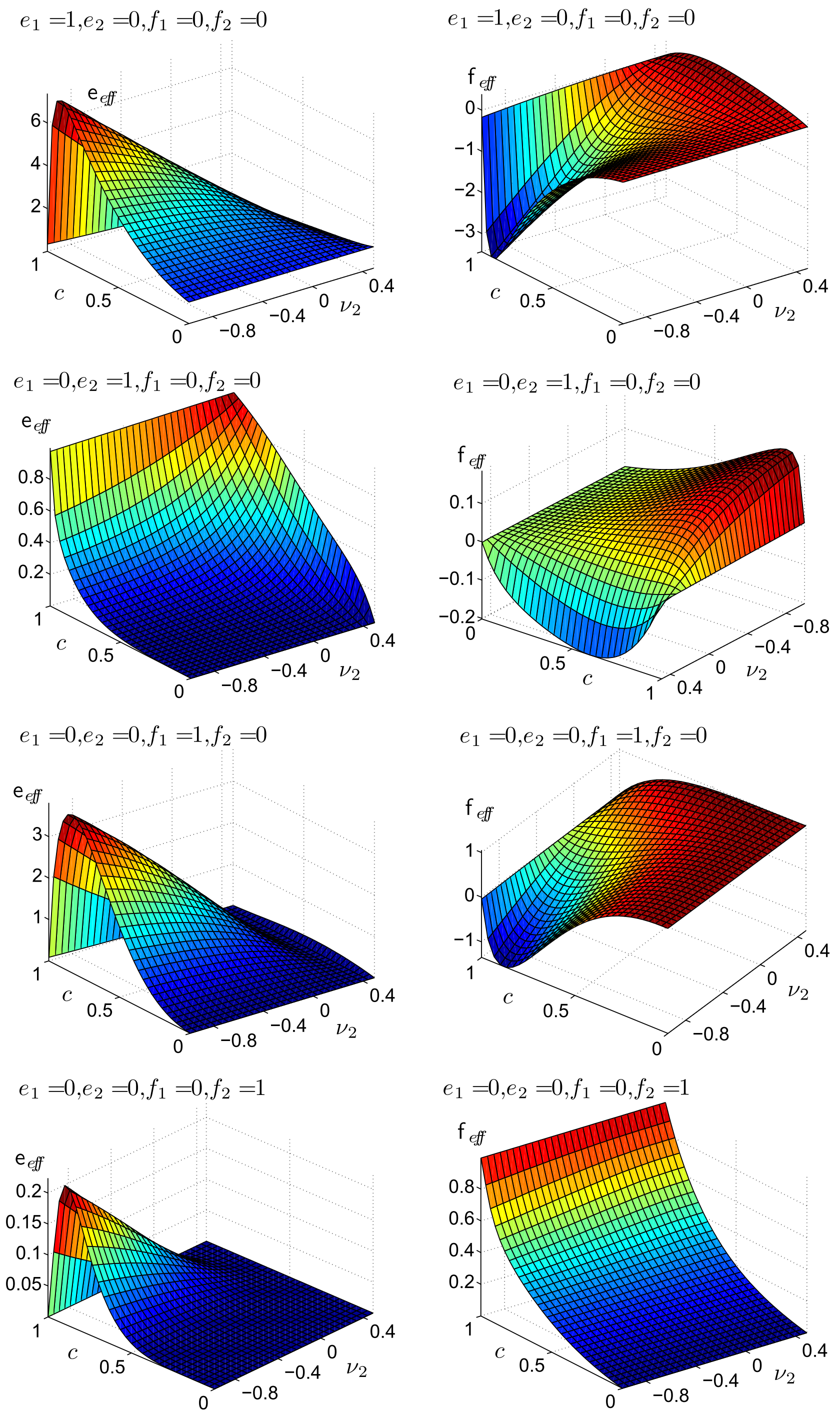


Fig. 5. Nonlinear effective elastic moduli in terms of the volume fraction c and the Poisson ratio ν_2 of the embedded inhomogeneity (or core). We have adopted the parameters $k_1 = 1$, $\nu_1 = 0.33$, $k_2 = 2$. Four couples of plots for e_{eff} and f_{eff} correspond to the nonlinear parameter of the structure ($e_1 = 1$, $e_2 = 0$, $f_1 = 0$, $f_2 = 0$), ($e_1 = 0$, $e_2 = 1$, $f_1 = 0$, $f_2 = 0$), ($e_1 = 0$, $e_2 = 0$, $f_1 = 1$, $f_2 = 0$) and ($e_1 = 0$, $e_2 = 0$, $f_1 = 0$, $f_2 = 1$). It is therefore possible to observe the effects of any nonlinearity source separately.

tive nonlinearities for a negative contrast $\log_{10}(k_2/k_1)$ (i.e., $k_2 \ll k_1$).

Moreover, in Fig. 5 we can find the results of a second analysis conducted to obtain the nonlinear effective elastic moduli in terms of the volume fraction c and the Poisson ratio ν_2 of the embedded inhomogeneity (or core). We have adopted the parameters $k_1 = 1$, $\nu_1 = 0.33$, $k_2 = 2$. As before, four couples of plots for e_{eff} and f_{eff} correspond to the nonlinear parameter of the structure ($\mathsf{e}_1 = 1$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 0$), ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 1$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 0$), ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 1$, $\mathsf{f}_2 = 0$) and ($\mathsf{e}_1 = 0$, $\mathsf{e}_2 = 0$, $\mathsf{f}_1 = 0$, $\mathsf{f}_2 = 1$). Also in this case we observe a very intriguing and complex mixing behavior of the nonlinear features. In particular we note that, when the nonlinearity is confined within the external shell, the nonlinear parameter e_{eff} exhibits a strong positive amplification effect for a negative Poisson ratio ν_2 of the core and, on the other hand, the nonlinear parameter f_{eff} exhibits a negative amplification effect for a negative Poisson ratio of the core. Moreover, when $\mathsf{e}_2 = 1$ we have a quite constant e_{eff} for different values of ν_2 and two intensification effects of f_{eff} (i.e., a negative peak for positive ν_2 and a positive peak for negative ν_2). Finally, when $\mathsf{f}_2 = 1$ we have a quite constant f_{eff} for different values of ν_2 and an intensification effect of e_{eff} (i.e., a positive peak for negative ν_2).

4. Applications and comparisons with earlier theories

In this section we show some applications of the general theory to specific cases of technological interest and we draw some comparisons with certain previous approximated theories.

In particular we analyse a nonlinear cylindrical shell with a void core, we find some simple results concerning the more specific case of a thin nonlinear tube or nanotube and, finally, we discuss the relations between the present approach and a previous one dealing with a homogenization scheme for a dispersion of nonlinear parallel cylinder in a linear matrix.

4.1. Nonlinear cylindrical shells with a void core

As first application of the general theory, we take into consideration the case of a nonlinear shell with a void core. It means that we

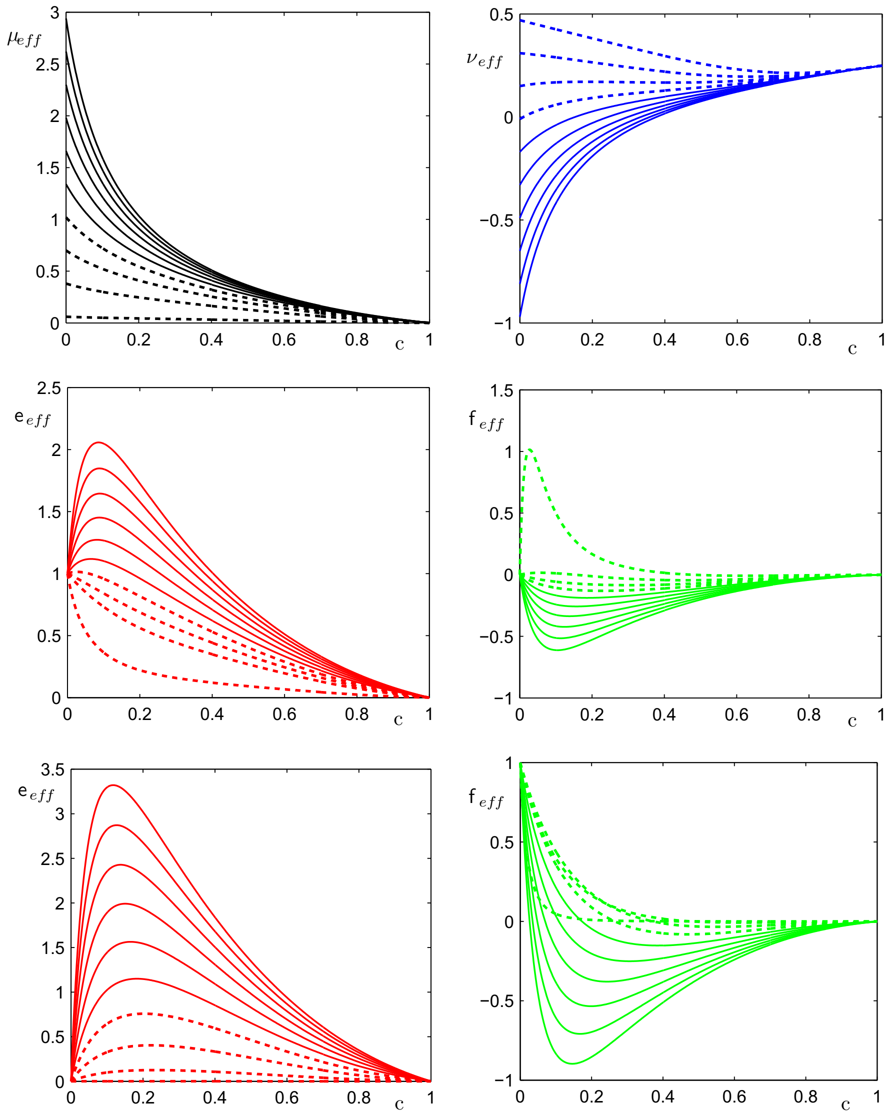


Fig. 6. Results for a void core embedded in the cylindrical structure with nonlinear matrix. We have adopted the parameters $k_1 = 1$, $0 < \mu_1 < 3k_1$ (or equivalently $-1 < \nu_1 < 1/2$) sampled through 10 equispaced values corresponding to lines in figures. In the first line one can find the linear results for μ_{eff} and ν_{eff} (dashed lines for positive matrix Poisson ratio and solid lines for negative matrix Poisson ratio). As for the nonlinear behavior in the second line e_{eff} and f_{eff} are represented for $\mathsf{e}_1 = 1$, $\mathsf{f}_1 = 0$ and, in the third line for $\mathsf{e}_1 = 0$, $\mathsf{f}_1 = 1$. The perforated matrix exhibits the nonlinear parameter not shown in the bulk material and also displays remarkable nonlinear intensification effects.

consider $k_2 = 0$, $\nu_2 = 0$, $\mathrm{e}_2 = 0$ and $\mathrm{f}_2 = 0$ throughout this section. On the other hand, within the shell (or matrix) we consider $k_1 = 1$ and we use ten equispaced values for $0 < \mu_1 < 3k_1$. When $0 < \mu_1 < k_1$ we have a positive Poisson ratio ν_1 and, conversely, when $k_1 < \mu_1 < 3k_1$ we obtain a negative value for the Poisson value ν_1 (it is evident by considering the relation $\nu_1 = (k_1 - \mu_1)/(2k_1)$). So, we can explore all the possibilities for the elasticity of the matrix. In Fig. 6 we show the results obtained with the general theory summarized in Eqs. (68)–(71). In the first line the linear results for μ_{eff} (black lines) and ν_{eff} (blue lines) are represented in terms of the void volume fraction c. Everywhere we have adopted dashed lines for results corresponding to the positive values of ν_1 and solid lines for the cases with $\nu_1 < 0$. It is interesting to observe the convergence of ν_{eff} to $1/4$ when the volume fraction c is approaching the value 1. This phenomenon is consistent with the universal behavior of the effective Poisson ratio of porous materials for high values of the porosity (Zimmerman, 1991, 1994; Christensen, 1993; Giordano, 2003). As for the nonlinear effective results, in the second line of Fig. 6, $\mathrm{e}_{\mathrm{eff}}$ (red lines) and $\mathrm{f}_{\mathrm{eff}}$ (green lines) are represented for the case with $\mathrm{e}_1 = 1$ and $\mathrm{f}_1 = 0$; similarly, in the third line, $\mathrm{e}_{\mathrm{eff}}$ (red lines) and $\mathrm{f}_{\mathrm{eff}}$ (green lines) are represented for the case with $\mathrm{e}_1 = 0$ and $\mathrm{f}_1 = 1$. We can observe two general interesting properties:

- In both cases we observe a nonlinear effective behavior which exhibits both nonlinear parameters $\mathrm{e}_{\mathrm{eff}} \neq 0$ and $\mathrm{f}_{\mathrm{eff}} \neq 0$ independently of the fact that $\mathrm{f}_1 = 0$ or $\mathrm{e}_1 = 0$. In other words, after the perforation, we notice the appearance of the nonlinear parameter not present in the original matrix (i.e., $\mathrm{f}_{\mathrm{eff}} \neq 0$ if $\mathrm{f}_1 = 0$, and $\mathrm{e}_{\mathrm{eff}} \neq 0$ if $\mathrm{e}_1 = 0$). This property can be exploited to create composite structures with specific desired nonlinear response.
- We also observe that for certain values of the volume fraction there is a remarkable intensification of the nonlinear effects. This phenomenon is even more pronounced for a negative Poisson ratio of the shell.

4.2. Thin nonlinear tubes or nanotubes

We consider thin tubes or nanotubes with a nonlinear elastic behavior. To be concrete we can think of single-walled or multi-walled nanotubes but the applications are not restricted to these cases. A typical composite structure is obtained by a dispersion of parallel nanotubes, embedded in a given matrix in order to develop reinforcing techniques and/or sensors designing. With the idea of implementing the multiscale methodology, it is first important to ideally substitute each embedded nanotube with an effective uniform cylinder. Then, as second step, we can apply standard homogenization techniques to obtain the overall behavior of the whole dispersion (see e.g., the next section for details). The first step of the above procedure can be performed by considering a composite cylinder with a void core and a thin shell: it means that we can perform the developments of the general equations (68)–(71) with $k_2 = 0$, $\nu_2 = 0$, $\mathrm{e}_2 = 0$, $\mathrm{f}_2 = 0$ and the volume fraction approaching the value $c = 1$ (thin structure). A long but straightforward calculation allows us to obtain the following results, which are valid up to the second order in the variable $1 - c$

$$\mu_{\mathrm{eff}} = \frac{k_1\mu_1}{2(k_1+\mu_1)}(1-c) + \frac{k_1\mu_1(\mu_1+2k_1)}{2(k_1+\mu_1)^2}(1-c)^2 + o\left((1-c)^2\right) \tag{72}$$

$$k_{\mathrm{eff}} = \frac{k_1\mu_1}{k_1+\mu_1}(1-c) + \frac{k_1^2\mu_1}{(k_1+\mu_1)^2}(1-c)^2 + o\left((1-c)^2\right) \tag{73}$$

$$\begin{aligned}\mathrm{e}_{\mathrm{eff}} = {} & \frac{3\mu_1(\mu_1^2+k_1^2)}{2(k_1+\mu_1)^3}\mathrm{e}_1(1-c) + \frac{3\mu_1^3}{(k_1+\mu_1)^3}\mathrm{f}_1(1-c) \\ & + \frac{3\mu_1(\mu_1^3+4k_1\mu_1^2+3k_1^3)}{2(k_1+\mu_1)^4}\mathrm{e}_1(1-c)^2 \\ & + \frac{3\mu_1^3(\mu_1+4k_1)}{(k_1+\mu_1)^4}\mathrm{f}_1(1-c)^2 + o\left((1-c)^2\right)\end{aligned} \tag{74}$$

$$\begin{aligned}\mathrm{f}_{\mathrm{eff}} = {} & -\frac{\mu_1^3}{2(k_1+\mu_1)^3}\mathrm{f}_1(1-c) - \frac{\mu_1(\mu_1^2+k_1^2)}{4(k_1+\mu_1)^3}\mathrm{e}_1(1-c) \\ & - \frac{\mu_1(3\mu_1^3+6k_1\mu_1^2+5k_1^3+2k_1^2\mu_1)}{4(k_1+\mu_1)^4}\mathrm{e}_1(1-c)^2 \\ & - \frac{3\mu_1^3(\mu_1+2k_1)}{2(k_1+\mu_1)^4}\mathrm{f}_1(1-c)^2 + o\left((1-c)^2\right)\end{aligned} \tag{75}$$

As already observed in the previous section, the nonlinear effective parameters $\mathrm{e}_{\mathrm{eff}}$ and $\mathrm{f}_{\mathrm{eff}}$ are influenced by both the nonlinear coefficients e_1 and f_1, thus confirming the strong coupling between the nonlinear properties.

4.3. Homogenization of dispersions of parallel fibres

In this last section we want to study the relations between the present general theory and previous homogenization schemes. To this aim, we briefly introduce an earlier result concerning the two-dimensional homogenization of a dispersion of circular nonlinear inhomogeneities embedded in a linear isotropic plane (Giordano, 2009; Giordano et al., 2008, 2009; Colombo and Giordano, 2011). The structure of this composite material is depicted in Fig. 7. The circular inhomogeneities are randomly and isotropically embedded into a linear matrix with elastic moduli μ_1 and k_1 (k_1 is always the two-dimensional version of the bulk modulus). To begin, we suppose that the volume fraction $c$of the embedded phase is small (dilute dispersion). If we identify the linear coefficients of the inhomogeneities by μ_2 and k_2 and their nonlinear constants by e_2 and f_2, then the stress–strain relation is given by Eq. (23) with $\alpha = 2$. The constitutive equation of the whole system is expressed in terms of the effective linear and nonlinear elastic moduli as in Eq. (24). In this case the effective linear elastic moduli are given by

$$\mu_{\mathrm{eff}} = \mu_1 + c\frac{\mu_2-\mu_1}{c+(1-c)\left[1+\frac{1}{2}\left(\frac{\mu_2}{\mu_1}-1\right)\frac{k_1+2\mu_1}{k_1+\mu_1}\right]} \tag{76}$$

$$k_{\mathrm{eff}} = k_1 + c\frac{k_2-k_1}{c+(1-c)\frac{\mu_1+k_2}{\mu_1+k_1}} \tag{77}$$

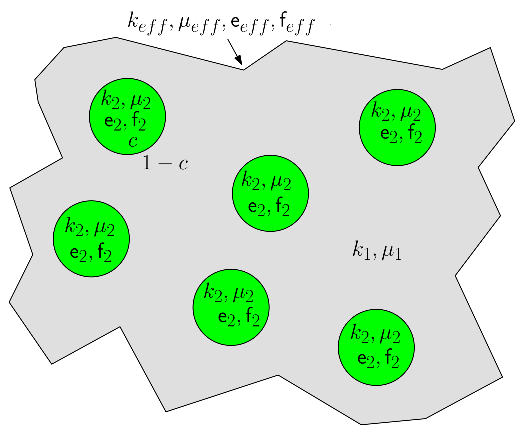


Fig. 7. Scheme of a dispersion of nonlinear circular inhomogeneities (with linear moduli μ_2 and k_2 and nonlinear constants e_2 and f_2) embedded into a linear matrix with elastic moduli μ_1 and k_1.

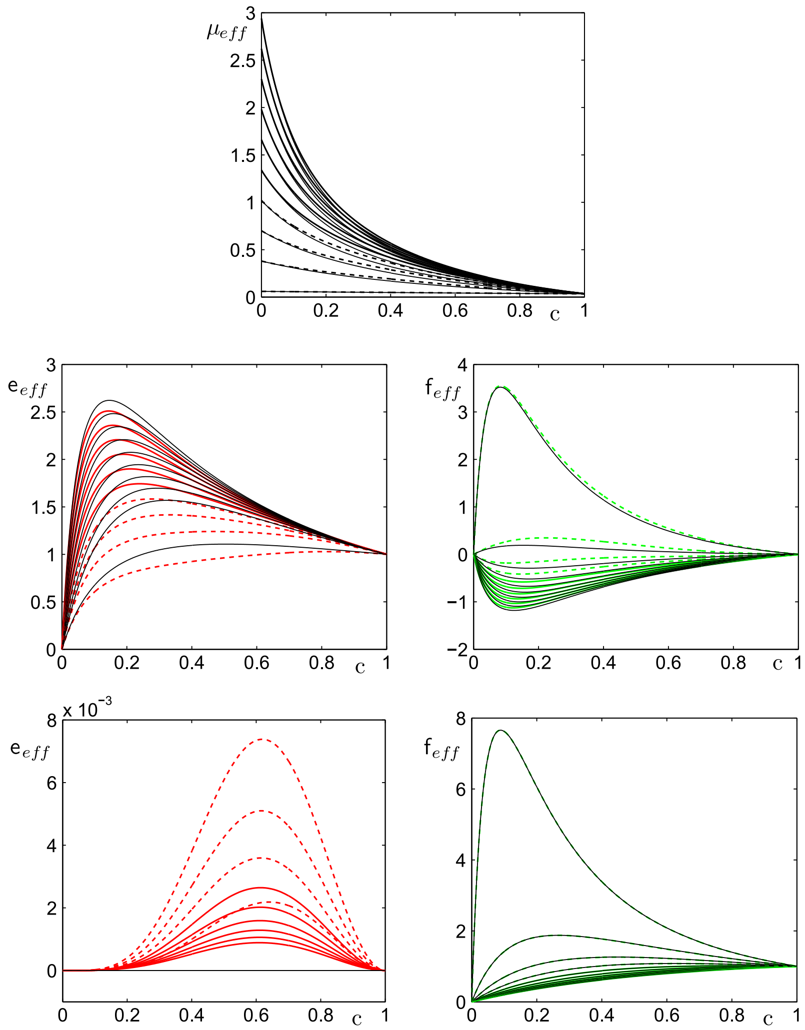


Fig. 8. Results for the composite structure with the linear matrix (inhomogeneity softer than the matrix): comparison of the general theory given in Eqs. (68)–(71) (coloured lines, dashed for positive matrix Poisson ratio and solid for negative matrix Poisson ratio) with the old theory given in Eqs. (76)–(79) (thin black lines). The first plot represents the linear result for μ_{eff}; the two couples of nonlinear results for e_{eff} and f_{eff} correspond to $\mathrm{e}_2 = 1$, $\mathrm{f}_2 = 0$ and $\mathrm{e}_2 = 0$, $\mathrm{f}_2 = 1$, respectively. All effective parameters have been represented versus the volume fraction c. We have adopted the parameters $k_1 = 1$, $0 < \mu_1 < 3k_1$ (or equivalently $-1 < \nu_1 < 1/2$) sampled through 10 equispaced values corresponding to lines in figures, $k_2 = 0.1$, $\nu_2 = 0.33$.

and the effective nonlinear elastic moduli by

$$\mathrm{e}_{\text{eff}} = \frac{\mathrm{e}_2 c}{\mathcal{L}^2}\left(1 - \frac{1-c}{\mathcal{L}+2\mathcal{M}}\frac{k_2-k_1}{\mu_1+k_1}\right) \tag{78}$$

$$\begin{aligned}\mathrm{f}_{\text{eff}} = &-\frac{c(1-c)(k_2-k_1)(\mathrm{e}_2+2\mathrm{f}_2)}{2(k_1+\mu_1)(\mathcal{L}+2\mathcal{M})^3} - \frac{\mathrm{e}_2 c}{6\mathcal{L}^2} + \frac{c(\mathrm{e}_2+2\mathrm{f}_2)}{2(\mathcal{L}+2\mathcal{M})^2} \\ &- \frac{\mathrm{e}_2 c}{3\mathcal{L}(\mathcal{L}+2\mathcal{M})} \\ &+ \frac{\mathrm{e}_2 c(1-c)\left[\mu_1(k_2-k_1)+(k_1+2\mu_1)(\mu_2-\mu_1)\right]}{6\mu_1(k_1+\mu_1)\mathcal{L}^2(\mathcal{L}+2\mathcal{M})}\end{aligned} \tag{79}$$

where we have defined the coefficients

$$\mathcal{L} = c + (1-c)\left[1 + \frac{1}{2}\frac{k_1+2\mu_1}{k_1+\mu_1}\left(\frac{\mu_2}{\mu_1}-1\right)\right] \tag{80}$$

$$\mathcal{M} = (1-c)\frac{1}{4(k_1+\mu_1)}\left[2k_2 - k_1\left(1+\frac{\mu_2}{\mu_1}\right) - 2(\mu_2-\mu_1)\right] \tag{81}$$

It is important also to consider the first order development of previous expressions with respect to the volume fraction c. We straightforwardly obtain

$$\mu_{\text{eff}} = \mu_1 + \frac{2\mu_1(\mu_2-\mu_1)(k_1+\mu_1)}{k_1\mu_2+2\mu_1\mu_2+k_1\mu_1}c + o(c^2) \tag{82}$$

$$k_{\text{eff}} = k_1 + \frac{(k_2-k_1)(k_1+\mu_1)}{\mu_1+k_2}c + o(c^2) \tag{83}$$

Similarly, for the effective nonlinear elastic moduli we get

$$\mathrm{f}_{\text{eff}} = \frac{(k_1+\mu_1)^3(\Xi_e \mathrm{e}_2 + \Xi_f \mathrm{f}_2)}{2(k_1\mu_2+2\mu_1\mu_2+k_1\mu_1)^2(\mu_1+k_2)^3}c + o(c^2) \tag{84}$$

$$\mathrm{e}_{\text{eff}} = \frac{4(k_1+\mu_1)^3\mu_1^2}{(k_1\mu_2+2\mu_1\mu_2+k_1\mu_1)^2(\mu_1+k_2)}\mathrm{e}_2 c + o(c^2) \tag{85}$$

where the coefficients Ξ_e and Ξ_f introduced in Eq. (84) are given by

$$\Xi_e = \left[k_1(\mu_1+\mu_2)+2\mu_1\mu_2\right]^2 - 4\mu_1^2(k_2+\mu_1)^2 \tag{86}$$

$$\Xi_f = 2(k_1\mu_2+2\mu_1\mu_2+k_1\mu_1)^2 \tag{87}$$

We draw a first comparison by analysing the first order expansions (for low values of the volume fraction c) of the general theory obtained in the present paper. The proposed nonlinear homogenization scheme yields the following results: from the linear point of

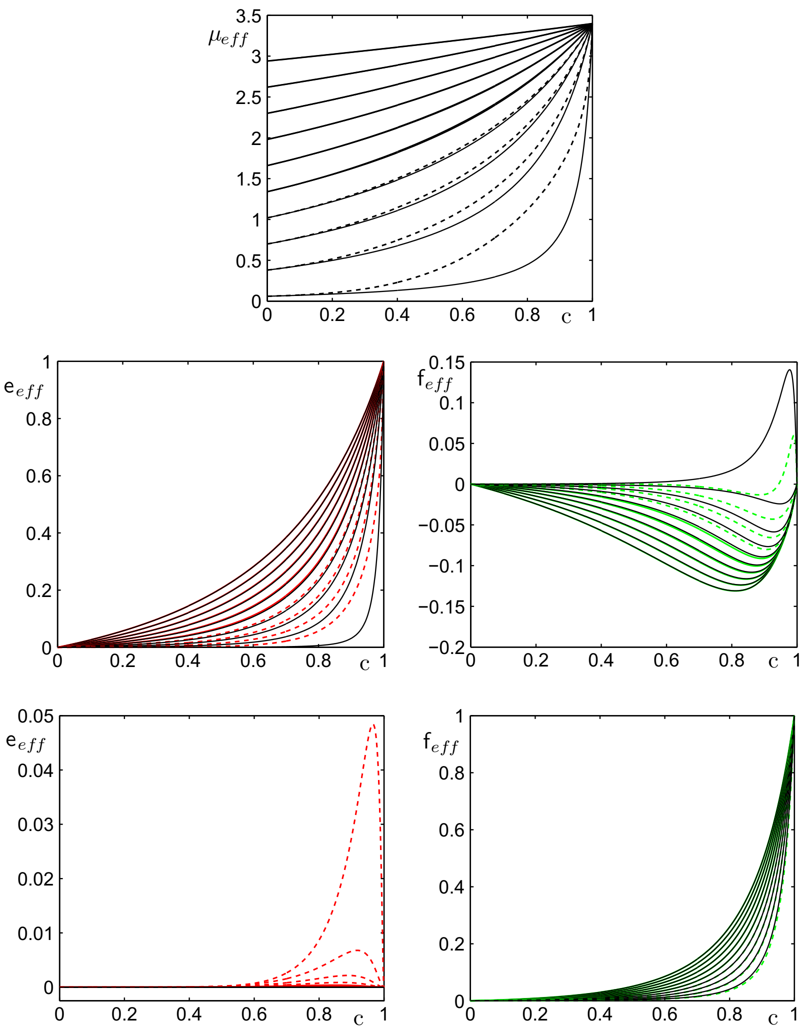


Fig. 9. Results for the composite structure with the linear matrix (inhomogeneity harder than the matrix): comparison of the general theory given in Eqs. (68)–(71) (coloured lines, dashed for positive matrix Poisson ratio and solid for negative matrix Poisson ratio) with the old theory given in Eqs. (76)–(79) (thin black lines). The first plot represents the linear result for μ_{eff}; the two couples of nonlinear results for e_{eff} and f_{eff} correspond to $e_2 = 1$, $f_2 = 0$ and $e_2 = 0$, $f_2 = 1$, respectively. All effective parameters have been represented versus the volume fraction c. We have adopted the parameters $k_1 = 1$, $0 < \mu_1 < 3k_1$ (or equivalently $-1 < \nu_1 < 1/2$) sampled through 10 equispaced values corresponding to lines in figures, $k_2 = 10$, $\nu_2 = 0.33$.

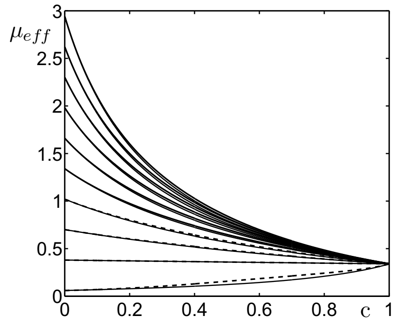
view, by using Eqs. (68) and (69) we obtain, for diluted structures ($c \ll 1$), the same expressions given in Eqs. (82) and (83). On the other hand, for the nonlinear part we can use Eqs. (70) and (71) and, therefore, we have the possibility to consider the nonlinear behavior of the matrix, not accounted for in Eqs. (84) and (85). Their form becomes

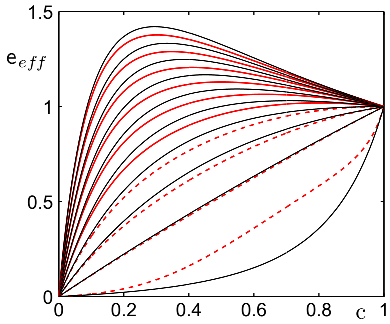
$$f_{eff} = f_1 + \frac{\frac{1}{2}(\mathcal{A}e_1 + \mathcal{B}f_1 + \mathcal{C}e_2 + \mathcal{D}f_2)c}{(k_1\mu_2 + 2\mu_1\mu_2 + k_1\mu_1)^2(\mu_1 + k_2)^3} + o(c^2) \quad (88)$$

$$e_{eff} = e_1 + \frac{(\mathcal{E}e_1 + \mathcal{F}f_1 + \mathcal{G}e_2)c}{(k_1\mu_2 + 2\mu_1\mu_2 + k_1\mu_1)^2(\mu_1 + k_2)} + o(c^2) \quad (89)$$

where the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ are reported in Appendix A. When the matrix is linear we have the coincidence of Eqs. (84) and (85) with Eqs. (88) and (89). In fact, in this case we have $e_1 = f_1 = 0$ and the following relation are satisfied: $\mathcal{C} = (k_1 + \mu_1)^3 \Xi_e$, $\mathcal{D} = (k_1 + \mu_1)^3 \Xi_f$ and $\mathcal{G} = 4\mu_1^2(k_1 + \mu_1)^3$ (see Appendix A for details).

A first general property can be therefore stated as follows. The two following results are exactly coincident: (i) the first order expansions of the effective medium theory for a dispersion of non linear inhomogeneity in a linear matrix; (ii) the first order expansions of the exact effective results for a composite cylinder (with a linear shell). We infer that the results obtained in the present paper for a composite cylinder can be also applied to the case of dispersions (at least for the dilute situation). The important point is that now we are able to take into consideration both the possible nonlinearity of the matrix and the nonlinear behavior of the embedded cylinders. To better explain this issue we show in Figs. 8–10 a comparison between the new results stated in Eqs. (68)–(71) (coloured lines) with the old theory given in Eqs. (76)–(79) (thin black lines). Clearly, in order to draw a coherent comparison we have adopted a linear matrix in our general solution described by (68)–(71). We have presented the effective nonlinear parameter on the complete range $0 < c < 1$ of the volume fraction for three different compressibility contrasts: $k_2 = 0.1k_1$ in Fig. 8, $k_2 = 10k_1$ in Fig. 9 and $k_2 = k_1$ in Fig. 10. We deduce an overall good agreement between the theories, except for e_{eff} when $e_2 = 0$. In fact, in this case with $e_2 = 0$ the old theory provides $e_{eff} = 0$ (see Eq. (78)) while the new formalism gives a result different from zero because of the strong coupling among the different nonlinear properties. However, we remark that the values of e_{eff} with $e_2 = 0$ are





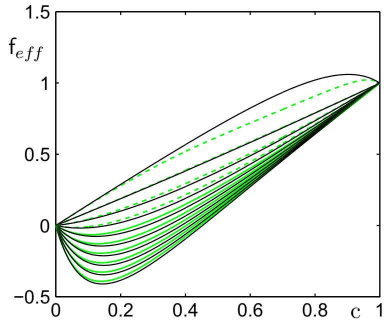


Fig. 10. Results for the composite structure with the linear matrix (without compressibility contrast between inhomogeneity and matrix): comparison of the general theory given in Eqs. (68)–(71) (coloured lines, dashed for positive matrix Poisson ratio and solid for negative matrix Poisson ratio) with the old theory given in Eqs. (76)–(79) (thin black lines). All effective parameters have been represented versus the volume fraction c. We have adopted the parameters $k_1 = 1$, $0 < \mu_1 < 3k_1$ (or equivalently $-1 < \nu_1 < 1/2$) sampled through 10 equispaced values corresponding to lines in figures, $k_2 = 1$, $\nu_2 = 0.33$, $e_2 = f_2 = 1$.

quite negligible also when calculated with the complete theory. In general, the small differences between the new predictions given in Eqs. (68)–(71) and the former theory resumed in Eqs. (76)–(79) can be explained as follows. New expressions are exact (up to the second order of nonlinearity) for the composite cylinder geometry and, therefore, they can not perfectly represent the random and isotropic distribution of fibers. On the other hand, the former result is an approximation specifically developed for the random structure. It is rather interesting to observe a good concordance between the different approaches.

To conclude, the procedure proposed in the present paper has been developed for a single composite cylinder (with two nonlinear phases), but it can be used (at least in approximate sense and for dilute structures) for dispersions of nonlinear cylinders embedded in a nonlinear matrix. According to the multiscale paradigm, we can further extend the applicability of the theory to the following situations:

- We may take into consideration a multi-shell composite cylinder composed of different nonlinear materials. Then, we can apply the present theory iteratively by starting with the homogenization of the core with the first shell. At the end we obtain a single uniform homogenized cylinder equivalent to the multi-shell original structure (the linear counterpart of this approach has been developed by Hervé and Zaoui (1995) and Hervé (2002)).
- We may also consider a dispersion of parallel cylinders (composed of many different nonlinear shells) embedded in a nonlinear matrix. To analyse this situation we can first use the previous point, homogenizing each multi-shell cylinder and, successively, we can obtain the effective behavior of the overall dispersion through the procedures discussed in the first part of the present section (a linear version of this idea has been applied by Stucu (1992)).

5. Conclusions

In this paper we have taken into account the problem of homogenizing a composite cylinder formed of a nonlinear elastic core embedded into a different nonlinear elastic shell. Each material has been assumed to be isotropic and, therefore, its behavior is represented by two linear elastic moduli (e.g., bulk and shear moduli) and two nonlinear coefficients (the so-called Landau coefficients). This is a minimal description of the SOEC and the TOEC, which is pertinent to the two-dimensional elasticity under plane strain conditions. We developed a homogenization procedure based on a nonlinear perturbation technique that allows us to obtain the effective linear and nonlinear behavior of the composite cylinder. All the effective properties (linear and nonlinear) can be found through the solution of a linear elastic problem, which has been approached by means of the complex variable method. As result we obtained the exact closed forms for the linear μ_{eff} and k_{eff} and nonlinear e_{eff} and f_{eff} effective elastic moduli, which are valid for any volume fraction of the core embedded in the external shell. We presented several applications of the general theory: we analysed a nonlinear cylindrical shell with a void core, we found some simple results concerning the case of a thin nonlinear tube or nanotube and, finally, we discussed the relations between the present approach and a previous one dealing with a homogenization scheme for a dispersion of nonlinear parallel cylinder in a linear matrix. In particular we have inferred that the exact theory for a single composite cylinder can be also used (with some approximations and for dilute structures) also for dispersions of nonlinear inhomogeneity in a different nonlinear matrix.

Appendix A. First order coefficients

A complete list of the coefficients entering Eqs. (88) and (89) is given below:

$$\begin{aligned}\mathcal{A} = {} & (\mu_1 + k_2)\Big(20\mu_1^5\mu_2 + 2\mu_1 k_1^3\mu_2^2 - 6k_2^2\mu_1^4 + k_1^4\mu_1^2 + 2k_1^4\mu_1\mu_2 \\ & - 2k_1^3\mu_1^3 - k_1^2\mu_1^4 + 4k_1\mu_1^5 - 8k_1^3k_2\mu_1\mu_2 - 12\mu_1^5k_2 + k_1^4\mu_2^2 \\ & - 6\mu_1^6 - 5k_1^2\mu_1^2\mu_2^2 - 12k_1\mu_1^3\mu_2^2 - 24\mu_1^3k_2\mu_2^2 + 44\mu_2\mu_1^4k_2 \\ & + 2k_1^2\mu_1^3k_2 - 6k_1^2\mu_1^3\mu_2 + 10k_1\mu_1^4k_2 - 6\mu_2^2\mu_1^2k_2^2 + 24\mu_1^3\mu_2k_2^2 \\ & - 4k_1^3k_2\mu_2^2 - 18k_1^2k_2\mu_2^2\mu_1 - 8k_1^2\mu_1^2k_2\mu_2 + 12k_1\mu_1^3k_2\mu_2 \\ & - 22k_1\mu_1^2k_2\mu_2^2 + 8k_1^2k_2^2\mu_2\mu_1 + 2k_1k_2^2\mu_2^2\mu_1 + 16k_1k_2^2\mu_2\mu_1^2 \\ & + 4k_1^2\mu_1^2k_2^2 + 6k_1\mu_1^3k_2^2 - 14\mu_1^4\mu_2^2 - 4k_1^3\mu_1^2k_2\Big) \qquad (A.1)\end{aligned}$$

$$\begin{aligned}\mathcal{B} = {} & -2(\mu_1 + k_2)^2\Big(-8\mu_2k_1\mu_1^2k_2 - 2\mu_1^2k_2\mu_2^2 - 12\mu_2\mu_1^3k_2 \\ & - 8k_1\mu_1k_2\mu_2^2 + 6\mu_1^4k_2 - 2k_1^2\mu_1^2k_2 - 4\mu_2k_1^2\mu_1k_2 \\ & - 2k_1^2k_2\mu_2^2 + 3k_1^3\mu_2^2 + 3k_1^3\mu_1^2 + 13k_1^2\mu_1\mu_2^2 + 4k_1\mu_1^3\mu_2 \\ & + 6k_1^3\mu_1\mu_2 + 6\mu_1^5 + k_1^2\mu_1^3 - 12\mu_1^4\mu_2 + 10\mu_1^3\mu_2^2 \\ & - 14k_1^2\mu_1^2\mu_2 + 16k_1\mu_1^2\mu_2^2\Big) \qquad (A.2)\end{aligned}$$

$$\begin{aligned}\mathcal{C} = {} & (k_1 + \mu_1)^3 \times (-2\mu_1k_2 + k_1\mu_1 - 2\mu_1^2 + k_1\mu_2 + 2\mu_1\mu_2) \\ & \times (2\mu_1k_2 + k_1\mu_1 + 2\mu_1^2 + k_1\mu_2 + 2\mu_1\mu_2) \qquad (A.3)\end{aligned}$$

$$\mathcal{D} = 2(k_1 + \mu_1)^3(k_1\mu_2 + k_1\mu_1 + 2\mu_1\mu_2)^2 \qquad (A.4)$$

$$\begin{aligned}\mathcal{E} = {} & 10\mu_1^3\mu_2^2 - 4k_1\mu_1^4 - 20\mu_1^4\mu_2 - k_1^3\mu_2^2 - k_1^3\mu_1^2 + 18\mu_1^2k_2\mu_2^2 \\ & - 24\mu_2\mu_1^3k_2 + 3k_1^2k_2\mu_2^2 - k_1^2\mu_1^2k_2 - 6k_1\mu_1^3k_2 \\ & - 4k_1\mu_1^2\mu_2^2 - 4k_1\mu_1^3\mu_2 - 4k_1^2\mu_1\mu_2^2 - 8k_1^2\mu_1^2\mu_2 \\ & - 2k_1^3\mu_1\mu_2 - 4\mu_2k_1\mu_1^2k_2 + 10k_1\mu_1k_2\mu_2^2 + 6\mu_1^5 \\ & + 6\mu_1^4k_2 - 2\mu_2k_1^2\mu_1k_2 \qquad (A.5)\end{aligned}$$

$$\mathcal{F} = 12\mu_1^2(\mu_2 - \mu_1)^2(\mu_1 + k_2) \qquad (A.6)$$

$$\mathcal{G} = 4\mu_1^2(k_1 + \mu_1)^3 \qquad (A.7)$$

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.ijsolstr.2013.08.017.

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