Infinite ergodicity in generalized geometric Brownian motions with nonlinear drift

Stefano Giordano [*]
University of Lille, CNRS, Centrale Lille, Univ. Polytechnique Hauts-de-France, UMR 8520 - IEMN - Institut d'Électronique, de Microélectronique et de Nanotechnologie, F-59000 Lille, France

Fabrizio Cleri [†]
University of Lille, Institut d'Électronique, de Microélectronique et de Nanotechnologie (IEMN CNRS UMR8520) and Departement de Physique, F-59652 Villeneuve d'Ascq, France

Ralf Blossey [‡]
University of Lille, Unité de Glycobiologie Structurale et Fonctionnelle (UGSF), CNRS UMR8576, F-59000 Lille, France

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Geometric Brownian motion is an exemplary stochastic processes obeying multiplicative noise, with widespread applications in several fields, e.g., in finance, in physics, and biology. The definition of the process depends crucially on the interpretation of the stochastic integrals which involves the discretization parameter α with $0 \leqslant \alpha \leqslant 1$, giving rise to the well-known special cases $\alpha = 0$ (Itô), $\alpha = 1/2$ (Fisk-Stratonovich), and $\alpha = 1$ (Hänggi-Klimontovich or anti-Itô). In this paper we study the asymptotic limits of the probability distribution functions of geometric Brownian motion and some related generalizations. We establish the conditions for the existence of normalizable asymptotic distributions depending on the discretization parameter α. Using the infinite ergodicity approach, recently applied to stochastic processes with multiplicative noise by E. Barkai and collaborators, we show how meaningful asymptotic results can be formulated in a transparent way.

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I. INTRODUCTION

Stochastic processes in the presence of multiplicative noise are a commonly encountered phenomenon in the sciences. In a general way, we consider a variable $x(t)$ which follows a stochastic differential equation. If the corresponding random term in the equation does not depend on the state of the system (i.e., on $x(t)$), we call it additive noise. On the other hand, if the random term depends on the state of the system $x(t)$, then the noise term is called multiplicative. A prime example of multiplicative noise from physics is the statistical theory of turbulence, where the energy cascade can be modeled by a Markov process at least down to the Taylor scale, which governs the intermediate scale of turbulent eddies [1]. The one-dimensional stochastic process across the scales, the Kolmogoroff-Obukhov theory (K62) [2] can be approximated by a geometric Brownian process. The geometric Brownian process is defined by a stochastic differential equation with the random term which is directly proportional to the state $x(t)$ of the system. A highly prominent example of geometric Brownian motion is the modeling of stock prices, notably with the celebrated Black-Scholes model of option pricing [3,4]. The relation between the stochastic behavior of financial markets and the turbulence cascade has been discussed in Ref. [5]. Multiplicative noises are also used for explaining the ballistic-to-diffusive transition of the heat propagation [6,7]. In this case, a chain of particles is considered with a multiplicative stochastic force field able to be energyconserving for each particle of the system. Random systems with multiplicative noise find applications also in cosmology and statistical field theory [8]. Moreover, stochastic differential equations with multiplicative noise have been largely studied to model the heterogeneous diffusion processes with and without resetting, with applications to several problems including the transport in heterogeneous materials, random fractals or amorphous systems, the analysis of financial time series, and the active motion of cells [9–15]. Finally, there are other applications in biology for which we cite as examples the stochastic firing of neurons [16,17], phenotypic variability and gene expression [18,19], and the motion of molecular motors [20], notably chromatin remodeling complexes acting on nucleosomes [21,22].

All these cases have in common that they deal with first-order stochastic differential equations of the form

$$\frac{dx}{dt} = h(x,t) + g(x,t)\xi(t), \tag{1}$$

where $h(x,t)$ is the drift term, $g(x,t)$ is the diffusion term, and the stochastic process $\xi(t)$ is a Gaussian noise with average value $E\{\xi(t)\} = 0$, and correlation $E\{\xi(t)\xi(\tau)\} = 2\delta(t-\tau)$. The stochastic differential equation has a well-defined meaning only if we declare the adopted interpretation of the stochastic integrals. To achieve this, we have to

[*]stefano.giordano@univ-lille.fr
[†]fabrizio.cleri@univ-lille.fr
[‡]ralf.blossey@univ-lille.fr

specify the discretization parameter α, defining the position of the point at which we calculate any integrated function in the small intervals of the adopted Riemann sum ($0 \leqslant \alpha \leqslant 1$) [23–26]. This integration theory includes the Itô ($\alpha = 0$) [27], the Fisk-Stratonovich ($\alpha = 1/2$) [28,29], and the Hänggi-Klimontovich or anti-Itô ($\alpha = 1$) [30,31] as particular cases (see also Ref. [32]). In fact, our above-mentioned example of the turbulent cascade, in which t is identified with the cascade scale, is commonly interpreted in the Fisk-Stratonovich sense, while the Black-Scholes stock market model is treated in the Itô-interpretation. For the heat conduction, it has been proved that all stochastic interpretations are equivalent [7].

The stochastic process can likewise be described with the help of the Fokker-Planck equation, a partial differential equation for the probability density function (PDF) $W(x,t)$ of the stochastic process given by [23–26,33]

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left[\left(h + 2\alpha g\frac{\partial g}{\partial x}\right)W\right] + \frac{\partial^2}{\partial x^2}(g^2 W), \quad (2)$$

where the first term represents the force-dependent drift, the second a noise-induced drift which explicitly depends on α, and the third the diffusion term generated by the noise. The noise-induced drift term is absent when $\partial g/\partial x = 0$, i.e., for purely additive noise. Thus, the choice of the stochastic calculus—the choice of the discretization parameter—is relevant only in the case of multiplicative noise. The theory can be generalized to take into consideration possible crosscorrelation of the noises [34,35]. The Fokker-Planck Eq. (2) can also be rewritten in the following useful form

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left\{-hW + g^{2\alpha}\frac{\partial}{\partial x}[g^{2(1-\alpha)}W]\right\}, \quad (3)$$

which is readily demonstrated by performing the derivatives.

In this paper we are interested in a full characterization of the probability distribution function (PDF) for geometric Brownian motion and some generalizations of this process. We will in particular consider the class of stochastic equations with simple algebraic nonlinearities for the drift and noise terms

$$\frac{dx}{dt} = H(t)x^n + G(t)x^m\xi(t) \quad (4)$$

since they will readily allow us to obtain analytic results. Specifically, we are interested in the conditions that guarantee the existence of a normalizable asymptotic long-time limit, or stationary PDFs, given by

$$W_{as}(x) = \lim_{t\to\infty} W(x,t). \quad (5)$$

That the existence of such PDFs is not generally guaranteed, and it indeed depends on the discretization parameter α, was recently shown by Barkai and collaborators [36–38] for certain cases we comment on below. The authors introduced the concept of infinite ergodicity in the discussion, which allowed them to define a procedure to extract meaningful physical quantities from these nonnormalizable distributions. In particular, it is possible to determine the asymptotic behavior (with time going to infinity) of the expected value of different physical observables. In statistical mechanics, these approaches are used when the potential energy is nonconfining and thus generates an infinite phase space (or infinite measure space [39]), hence the name infinite ergodicity. In Ref. [37], the case of geometric Brownian motion was explicitly excluded from the discussion, so we extend its analysis in the present work. More specifically, we consider a geometric Brownian motion, $m = 1$ in Eq. (4), with a nonlinear drift ($n \neq 1$) and then we introduce a generalization with a nonlinear diffusion term ($m \neq 1$).

The structure of the paper is as follows. In Sec. II, we introduce the geometric Brownian motion with timevarying and linear drift and diffusion terms. Here we obtain a generalized log-normal distribution. In Sec. III, we introduce a nonlinear drift term in the geometric Brownian motion stochastic equation, and we investigate the existence of normalizable asymptotic densities as defined in Eq. (5). In Sec. IV, we discuss the concept of infinite ergodicity by considering a simple overdamped system taken from statistical mechanics. We then apply this concept to the geometric Brownian motion with nonlinear drift term in Sec. V. To conclude, we generalize our approach for systems with an algebraic nonlinear diffusion term in Sec. VI.

II. TIME-VARYING GEOMETRIC BROWNIAN MOTION

We initially focus on geometric Brownian motion where the functions $h(x,t)$ and $g(x,t)$ are proportional to x [40–42]. Thus, we consider the time-varying geometric Brownian motion characterized by the stochastic Eq. (4) with $n = m = 1$,

$$\frac{dx}{dt} = H(t)x + G(t)x\xi(t), \quad (6)$$

where $H(t)$ and $G(t)$ are two arbitrary time-dependent functions. The Fokker-Planck Eqs. (2) and (3) can be written in this case as

$$\frac{\partial W}{\partial t} = -(H + 2\alpha G^2)\frac{\partial}{\partial x}(xW) + G^2\frac{\partial^2}{\partial x^2}(x^2 W), \quad (7)$$

and

$$\frac{\partial W}{\partial t} = -H\frac{\partial}{\partial x}(xW) + G^2\frac{\partial}{\partial x}\left\{x^{2\alpha}\frac{\partial}{\partial x}[x^{2(1-\alpha)}W]\right\}. \quad (8)$$

We are interested in finding the general solution of these equations for arbitrary functions $H(t)$ and $G(t)$. The driftless case $H(t) = 0$ and with constant $G(t) \equiv G_0$ is described by a log-normal distribution [40]

$$f_{\mathbf{x}}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}, \quad (9)$$

defined on the positive real line $x > 0$, with suitable real parameters σ and μ that define the shape of the distribution. The log function always refers to the natural logarithm. The first-order and second-order expectation values are given by the expressions

$$E\{\mathbf{x}\} = e^{\mu+\frac{\sigma^2}{2}}, \quad E\{\mathbf{x}^2\} = e^{2\mu+2\sigma^2}, \quad (10)$$

and the variance is given by

$$\sigma_{\mathbf{x}}^2 = E\{\mathbf{x}^2\} - E\{\mathbf{x}\}^2 = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}. \quad (11)$$

From Eq. (10), we deduce the parameters μ and σ^2 as function of the expectation values as

$$\mu = \log \frac{E\{\mathbf{x}\}^2}{\sqrt{E\{\mathbf{x}^2\}}}, \quad \sigma^2 = \log \frac{E\{\mathbf{x}^2\}}{E\{\mathbf{x}\}^2}. \tag{12}$$

We now assume that the solution of the Fokker-Planck equation also has a log-normal form for arbitrary functions $H(t)$ and $G(t)$. This leads to the following evolution equations for the expectation values:

$$\frac{dE\{\mathbf{x}\}}{dt} = [H(t) + 2\alpha G^2(t)]E\{\mathbf{x}\}, \tag{13}$$

$$\frac{dE\{\mathbf{x}^2\}}{dt} = 2[H(t) + (2\alpha + 1)G^2(t)]E\{\mathbf{x}^2\}. \tag{14}$$

These equations were obtained by multiplying the Fokker-Planck equation by x and by x^2 and integrating the results on the interval $(0, \infty)$. An integration by parts eventually leads to Eqs. (13) and (14). These differential equations can be solved to obtain

$$E\{\mathbf{x}\} = \mu_0 e^{\int_0^t [H(u)+2\alpha G^2(u)]du}, \tag{15}$$

$$E\{\mathbf{x}^2\} = \left(\mu_0^2 + \sigma_0^2\right) e^{2\int_0^t [H(u)+(2\alpha+1)G^2(u)]du}, \tag{16}$$

where μ_0 and σ_0^2 are the average value and the variance of $\mathbf{x}$ for $t = 0$, respectively. Substituting Eqs. (15) and (16) into Eq. (12), we get

$$\mu = \frac{1}{2} \log \frac{\mu_0^4}{\mu_0^2 + \sigma_0^2} + \int_0^t [H(u) + (2\alpha - 1)G^2(u)]du, \tag{17}$$

$$\sigma^2 = \log \frac{\mu_0^2 + \sigma_0^2}{\mu_0^2} + 2\int_0^t G^2(u)du. \tag{18}$$

In particular, if $\sigma_0 = 0$, we have

$$\mu = \log \mu_0 + \int_0^t [H(u) + (2\alpha - 1)G^2(u)]du, \tag{19}$$

$$\sigma^2 = 2\int_0^t G^2(u)du. \tag{20}$$

In order to demonstrate that the corresponding log-normal distribution really is the exact solution of our problem, the Fokker-Planck equation in Eq. (7) or Eq. (8), with the initial condition $W(x, 0) = \delta(x - \mu_0)$, must be solved by the trial density

$$W(x,t) = \frac{\exp\left\{-\frac{\left[\log\frac{x}{\mu_0} - \int_0^t [H(u)+(2\alpha-1)G^2(u)]du\right]^2}{4\int_0^t G^2(u)du}\right\}}{2x\sqrt{\pi \int_0^t G^2(u)du}}, \tag{21}$$

which follows from Eq. (9) combined with Eqs. (19) and (20). This can be verified by a lengthy but straightforward calculation, as discussed in Appendix A. Our result in Eq. (21) therefore generalizes the log-normal solution to the time-varying case independent of the interpretation of the stochastic integration rule ($0 \leqslant \alpha \leqslant 1$). We remark that the obtained solution automatically satisfies the reflecting boundary condition at $x = 0$.

While we have found a completely general expression for the PDF $W(x, t)$, a proper normalizable long-time limit for $t \to \infty$ of the PDF does not always exist, depending on the form of $H(u)$ and $G(u)$. A normalizable stationary PDF exists only if the integrals $\int_0^t H(u)du$ and $\int_0^t G^2(u)du$ converge to finite values for $t \to \infty$. For example, if we take constant values for these functions, $H(u) = H_0$ and $G(u) = G_0$, the density is given by

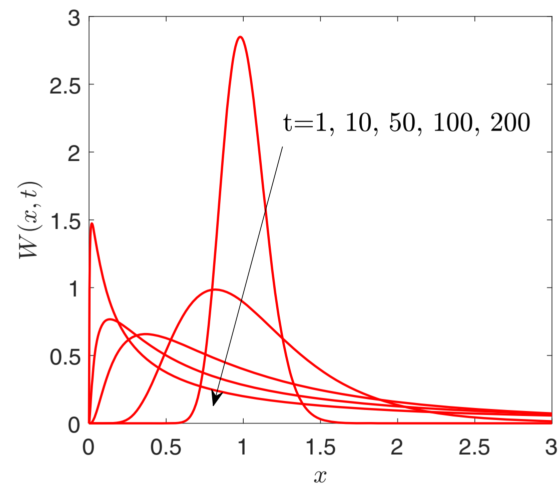

FIG. 1. Example of log-normal distribution evolution. We implemented Eq. (22) with the parameters $H_0 = 0$, $G_0 = 1/10$, $\mu_0 = 1$, and $\alpha = 1/2$.

$$W(x,t) = \frac{\exp\left\{-\frac{\left[\log\frac{x}{\mu_0} - H_0 t - (2\alpha-1)G_0^2 t\right]^2}{4G_0^2 t}\right\}}{2xG_0\sqrt{\pi t}}, \tag{22}$$

which does not converge to an asymptotic or equilibrium distribution (see Fig. 1). This observation is the starting point for our following discussion.

III. DRIFT EFFECT ON GEOMETRIC BROWNIAN MOTION

The result from the previous section leads us to investigate whether a suitable nonlinear drift term can generate an asymptotic equilibrium density for a constant $G(t) = G_0$. Thus, we now consider the stochastic differential equation

$$\frac{dx}{dt} = h(x) + G_0 x \xi(t), \tag{23}$$

where the drift term $h(x)$ is for the moment left unspecified. As before, we can associate the following Fokker-Planck equation, governing the evolution of the density $W(x, t)$

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}(hW) + G_0^2 \frac{\partial}{\partial x}\left\{x^{2\alpha}\frac{\partial}{\partial x}[x^{2(1-\alpha)}W]\right\}. \tag{24}$$

The asymptotic solution of this Fokker-Planck equation, $W_{as}(x) = \lim_{t\to\infty} W(x, t)$, fulfills the equation

$$0 = -(hW_{as}) + G_0^2 x^{2\alpha} \frac{d}{dx}[x^{2(1-\alpha)}W_{as}]. \tag{25}$$

Introducing $\Theta(x) = x^{2(1-\alpha)}W_{as}(x)$, we have the simpler equation

$$\frac{d\Theta(x)}{dx} = \frac{h(x)}{G_0^2 x^2}\Theta(x), \tag{26}$$

whose solution is

$$\Theta(x) = \exp\left(\int \frac{h(x)}{G_0^2 x^2} dx\right). \tag{27}$$

Expressed in terms of the stationary PDF $W_{as}(x)$ we have

$$W_{as}(x) = \frac{K}{x^{2(1-\alpha)}} \exp\left(\int \frac{h(x)}{G_0^2 x^2} dx\right), \tag{28}$$

where we introduced a normalization constant K. Interestingly, this expression is reminiscent of the Pope-Ching result (for $\alpha = 1/2$), stating a relationship between the PDF of any stationary process and the expectations of time derivatives of the state of the system [43]. More precisely, this result indicates that both the expectations of the time derivative squared and of the second time derivative define the PDF shape (see Eq. (7) in Ref. [43]). Interestingly, this result has been used to better understand turbulent flows data and has been thoroughly discussed in Ref. [44].

The case of a linear drift term $h(x) = H_0 x$ leads back to the geometric Brownian motion already studied in Sec. II, for which we have already seen that there is no asymptotic solution. If $h(x) = H_0 x$, Eq. (28) immediately gives

$$W_{as}(x) = K x^{-2(1-\alpha)+H_0/G_0^2}, \tag{29}$$

and it is readily seen that the integral $\int_0^\infty W_{as}(x)dx$ cannot converge. In fact, we know that $\int_1^\infty x^{-k}dx$ is convergent for $k > 1$, and that $\int_0^1 x^{-k}dx$ is convergent for $k < 1$, therefore it is impossible to have values of $2(1-\alpha) - H_0/G_0^2$ that render the integral $\int_0^\infty W_{as}(x)dx$ convergent for $x \to 0$ and $x \to \infty$ at the same time.

It is therefore of interest to investigate whether different nonlinear forcing terms are able to generate a normalizable asymptotic density and consider

$$h(x) = -H_0 x^n, \tag{30}$$

where n is a real number. In case x can be considered as the spatial coordinate of a particle, the drift term corresponds to the ratio of the force and the friction coefficient, with the force being the gradient of a potential. For the case of stock option prices or the turbulent cascade mentioned in the Introduction, the terms correspond to empirical nonlinearities derived from mapping to experimental or numerical data. The stochastic differential equation is now given by

$$\frac{dx}{dt} = -H_0 x^n + G_0 x \xi(t). \tag{31}$$

In this case, Eq. (28) yields

$$W_{as}(x) = \frac{K}{x^{2(1-\alpha)}} \exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right), \tag{32}$$

where $n \neq 1$. In order to have a normalized asymptotic density, the inverse of the constant K must be given by the integral

$$\frac{1}{K} = \int_0^{+\infty} \frac{1}{x^{2(1-\alpha)}} \exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right)dx, \tag{33}$$

the convergence of which must be carefully inspected. Since the integrand consists of the product of an algebraic and an exponential function, there arise two sets of conditions that can ensure convergence of the integral on the right hand side of Eq. (33).

(i) The term $x^{-2(1-\alpha)}$ is convergent for $x \to 0$ when $2(1-\alpha) < 1$, or $\alpha > 1/2$. In this case the exponential term must ensure the convergence for $x \to \infty$, which is the case if $H_0 > 0$ and $n-1 > 0$. Indeed, in this case, $\exp(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}) \to 0$ when $x \to \infty$. Finally, the integral in Eq. (33) is convergent if

$$\alpha > \tfrac{1}{2}, \;\; H_0 > 0, \;\text{and}\; n-1 > 0. \tag{34}$$

(ii) The term $x^{-2(1-\alpha)}$ ensures the convergence for $x \to \infty$ if $2(1-\alpha) > 1$, or $\alpha < 1/2$. So, the exponential must handle the convergence for $x \to 0$. This is possible if $H_0 < 0$ and $n-1 < 0$. Indeed, in this case, $\exp(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}) \to 0$ when $x \to 0$. Hence, the integral in Eq. (33) is also convergent if

$$\alpha < \tfrac{1}{2}, \;\; H_0 < 0, \;\text{and}\; n-1 < 0. \tag{35}$$

Within these two complementary regions of convergence we have the conditions that

$$G_0^2\frac{n-1}{H_0} > 0, \;\text{and}\; \frac{2\alpha-1}{n-1} > 0. \tag{36}$$

An important finding is that we have found nonlinear drift terms that are in fact able to generate an asymptotic equilibrium density even when $G(t) = G_0$. It is interesting to notice that, however, for $\alpha = 1/2$, the often invoked Fisk-Stratonovich case, the convergence condition cannot be fulfilled, and therefore we cannot use the Fisk Stratonovich interpretation of the stochastic calculus for Eq. (29) if the asymptotic density must remain normalizable.

Next we will try to obtain an explicit expression for the integral Eq. (33). To this aim, we introduce the substitution

$$t = \frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1} \Leftrightarrow x = \left(\frac{n-1}{H_0}G_0^2\right)^{\frac{1}{n-1}} t^{\frac{1}{n-1}}, \tag{37}$$

which leads to

$$\frac{1}{K} = \frac{1}{|n-1|}\left(\frac{n-1}{H_0}G_0^2\right)^{\frac{2\alpha-1}{n-1}} \int_0^{+\infty} t^{\frac{2\alpha-1}{n-1}-1} e^{-t} dt. \tag{38}$$

Here, we have included the term $|n-1|$ for the following reason: if $n-1 > 0$, when $x \to 0$, we have $t \to 0$, and when $x \to \infty$, we have $t \to \infty$. Conversely, if $n-1 < 0$, when $x \to 0$, we have $t \to \infty$, and when $x \to \infty$, we have $t \to 0$. Hence, the order of integration changes depending on the sign of $n-1$. The integral in Eq. (38) is of the form of the Gamma function, so that by using its definition [45–47]

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{39}$$

we finally obtain from Eq. (38) the result

$$\frac{1}{K} = \frac{1}{|n-1|}\left(\frac{n-1}{H_0}G_0^2\right)^{\frac{2\alpha-1}{n-1}} \Gamma\left(\frac{2\alpha-1}{n-1}\right). \tag{40}$$

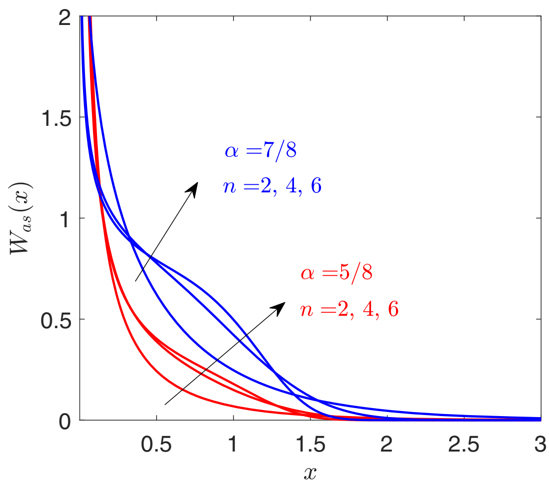

FIG. 2. Example of asymptotic distributions in the anti-Itô-side region defined by Eq. (34). We implemented Eq. (41) with the parameters $H_0 = 3/2$ and $G_0 = 1$.

Although we have shown that the integral is convergent in both the anti-Itô-side and Itô-side regions defined by Eqs. (34) and (35), it is also important to verify that the solutions of the corresponding stochastic differential equations do not present blow-up or explosion phenomena. This point is proved in Appendix B, where we also briefly introduce the concept of blow-up in finite time for ordinary and stochastic equations.

Anyway, the asymptotic density reads as

$$W_{as}(x) = \frac{|n-1|}{\left(\frac{n-1}{H_0}G_0^2\right)^{\frac{2\alpha-1}{n-1}}\Gamma\left(\frac{2\alpha-1}{n-1}\right)}\frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right)}{x^{2(1-\alpha)}}, \tag{41}$$

which is valid when the conditions in Eq. (36) are fulfilled. In Figs. 2 and 3 we show the shape of the asymptotic density for different values of the parameters in the anti-Itô-side and Itô-side regions, defined by Eqs. (34) and (35), respectively. We note that the densities are singular for $x = 0$ in the anti-Itô-side region while they are regular everywhere for the Itô-side region.

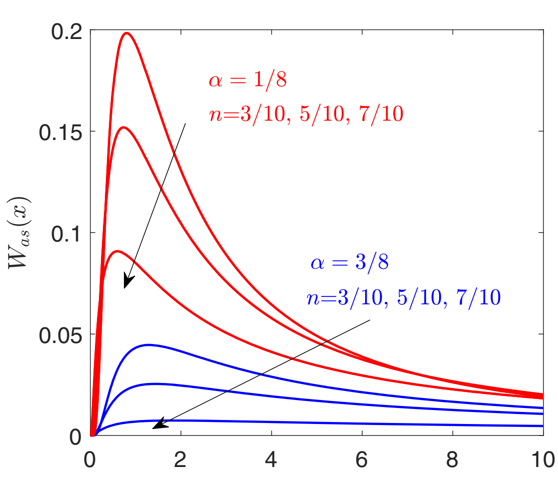

FIG. 3. Example of asymptotic distributions in the Itô-side region defined by Eq. (35). We implemented Eq. (41) with the parameters $H_0 = -3/2$ and $G_0 = 1$.

The next step consists of studying whether the asymptotic function also has a meaning if the normalization is not possible, e.g., in the Fisk-Stratonovich case, $\alpha = 1/2$. Since the concept of infinite ergodicity has only recently been brought into physics [36–38], before studying the previous problem for the case of geometric Brownian motion, we study it for a simpler example from statistical mechanics that will allow us to better introduce the concept of infinite ergodicity.

IV. INFINITE ERGODIC THEORY IN STATISTICAL MECHANICS

In order to introduce the concept of infinite ergodicity into our discussion, in this section we consider a simpler statistical mechanics system, following Refs. [36,38]. Let us consider a particle of mass m undergoing a one-dimensional overdamped stochastic motion under the effect of a potential energy $V(x)$. The Langevin equation reads as

$$\frac{dx}{dt} = -\frac{1}{m\gamma}\frac{dV}{dx} + \sqrt{\frac{k_BT}{m\gamma}}\xi(t), \tag{42}$$

where γ is the friction coefficient for unit mass, k_B is the Boltzmann constant, T is the temperature, and $\xi(t)$ is the noise with the same properties as given above; note that in this case the noise is simply additive and not multiplicative. The Fokker-Planck or Smoluchowski equation for the density $W(x,t)$ is given by [48,49]

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{m\gamma}\frac{\mathrm{d}V}{\mathrm{d}x}W + \frac{k_BT}{m\gamma}\frac{\partial W}{\partial x}\right). \tag{43}$$

As before, we search for the equilibrium distribution $W_{as}(x)$ through the equation

$$0 = \frac{1}{m\gamma}\frac{dV}{dx}W_{as} + \frac{k_BT}{m\gamma}\frac{dW_{as}}{\mathrm{d}x}, \tag{44}$$

which is solved by the Boltzmann distribution

$$W_{as}(x) = Ke^{-\frac{1}{k_BT}V(x)}, \tag{45}$$

where $1/K$ is the classical partition function. This density makes sense only if the partition function $\int \exp(-\frac{1}{k_BT}V)dx$ converges in the region of interest. Typically, problems of nonconvergence often emerge when nonconfining potentials are used. We try here to give a physical meaning to the obtained asymptotic density even when it is not normalizable. If $V(x) = 0$, of course, the partition function is not convergent but the general solution of the Fokker-Planck equation (which then reduces to the heat equation) is known. It can be obtained through an Ornstein-Uhlenbeck process with a quadratic potential energy $V(x) = \frac{1}{2}kx^2$ where $k \to 0$ [50–52], or more simply by invoking the Gaussian propagator for the free diffusion [23–26]. Anyway, the result is

$$W(x,t) = \sqrt{\frac{m\gamma}{4\pi k_BTt}}\exp\left[-\frac{m\gamma}{4k_BTt}(x-\mu_0)^2\right] \tag{46}$$

with the initial density $W(x,0) = \delta(x-\mu_0)$. We can now imagine that for long times, in a case with a nonconvergent partition function, the PDF evolution is given by a combination of Eqs. (45) and (46):

$$W(x,t) \underset{t\to\infty}{\sim} \sqrt{\frac{m\gamma}{4\pi k_B T t}} e^{-\frac{1}{k_B T}V(x)} e^{-\frac{m\gamma}{4k_B T t}(x-\mu_0)^2}, \tag{47}$$

or rather

$$W(x,t) \underset{t\to\infty}{\sim} \sqrt{\frac{m\gamma}{4\pi k_B T t}} e^{-\frac{1}{k_B T}V(x)}, \tag{48}$$

where we have used the property $\lim_{t\to\infty} e^{-\frac{m\gamma}{4k_B T t}(x-\mu_0)^2} = 1$. Let us also consider that $V(x) \to 0$ for $x \to +\infty$ and/or $x \to -\infty$ in correspondence with the nonconfining regions of the potential energy. In these regions we have a diffusive behavior of the system since the drift is negligible. The explored phase space is therefore infinite. In order to verify the conjecture in Eq. (48) we have to demonstrate that the same expression is the solution of the Fokker-Planck equation for long times. From Eq. (48) the left hand side of Eq. (43) is obtained as

$$\frac{\partial W(x,t)}{\partial t} \underset{t\to\infty}{\sim} -\sqrt{\frac{m\gamma}{16\pi k_B T t^3}} e^{-\frac{1}{k_B T}V(x)}. \tag{49}$$

Moreover, it is verified that the right hand side of Eq. (43) is exactly zero when calculated with Eq. (48). This indeed proves what is sought, since the term $\frac{\partial W(x,t)}{\partial t}$ goes to zero as $t^{-3/2}$, which is much faster than $t^{-1/2}$ (the leading term when $t \to \infty$), and is therefore negligible for long times, where we search for the solution. The remarkable point is that, from Eq. (48), we can write

$$\lim_{t\to\infty} \sqrt{\frac{4\pi k_B T t}{m\gamma}} W(x,t) = e^{-\frac{1}{k_B T}V(x)}, \tag{50}$$

a result also giving an important role to the Boltzmann exponential for the case with divergent partition function.

Furthermore, we can define an observable $\mathcal{O}(x)$ and introduce its ensemble average as

$$\langle \mathcal{O}(x)\rangle(t) = \int_{-\infty}^{+\infty} \mathcal{O}(x) W(x,t) dx. \tag{51}$$

From Eq. (50), we can write

$$\lim_{t\to\infty} \sqrt{\frac{4\pi k_B T t}{m\gamma}} \langle \mathcal{O}(x)\rangle(t) = \int_{-\infty}^{+\infty} \mathcal{O}(x) e^{-\frac{1}{k_B T}V(x)} dx, \tag{52}$$

which represents the infinite ergodicity property and, again, restores a role for the Boltzmann exponential factor also for the case with a divergent PDF. It means that the nonconfining potential generates an infinite phase space (hence the term infinity ergodicity) explored by a drift-diffusion process, whose asymptotic properties are described by Eq. (52). It is important to remark that the asymptotic behavior stated in Eq. (52) is correct only if the improper integral in the right hand side is convergent. Of course, several potential energies lead to the divergence of this integral and therefore the property is not valid in these cases.

Then, for completeness, we discuss the convergence of the integral in Eq. (52) for some forms of potential energy. To begin we suppose that $\mathcal{O}(x) = V(x)$ and we consider three cases.

(i) The potential energy is nonconfining on the left: $\lim_{x\to-\infty} V(x) = 0$ and $\lim_{x\to+\infty} V(x) = +\infty$. In this case the convergence of the integral in Eq. (52) is handled by $V(x)$ for $x \to -\infty$, and by $e^{-\frac{1}{k_B T}V(x)}$ for $x \to +\infty$, see Fig. 4(a).

(ii) The potential energy is nonconfining on the right: $\lim_{x\to-\infty} V(x) = +\infty$ and $\lim_{x\to+\infty} V(x) = 0$. In this case the convergence of the integral in Eq. (52) is handled by $e^{-\frac{1}{k_B T}V(x)}$ for $x \to -\infty$, and by $V(x)$ for $x \to +\infty$, see Fig. 4(b).

(iii) The potential energy is nonconfining on both the left and the right: $\lim_{x\to-\infty} V(x) = 0$ and $\lim_{x\to+\infty} V(x) = 0$. In this case the convergence of the integral in Eq. (52) is handled by $V(x)$ for both $x \to -\infty$ and $x \to +\infty$, see Fig. 4(c).

The same discussion remains valid if we consider the force as observable, namely $\mathcal{O}(x) = -dV(x)/dx$. If, as an example, we consider a potential energy nonconfining on the right (with $\lim_{x\to-\infty} V(x) = +\infty$ and $\lim_{x\to+\infty} V(x) = 0$), we can write

$$\begin{aligned}\lim_{t\to\infty} \sqrt{\frac{4\pi k_B T t}{m\gamma}} \left\langle -\frac{dV}{dx}\right\rangle &= \int_{-\infty}^{+\infty} -\frac{dV}{dx} e^{-\frac{1}{k_B T}V} dx \\ &= k_B T \int_{-\infty}^{+\infty} \frac{d}{dx}\left(e^{-\frac{1}{k_B T}V}\right) dx \\ &= k_B T\left[e^{-\frac{1}{k_B T}V(+\infty)} - e^{-\frac{1}{k_B T}V(-\infty)}\right] = k_B T,\end{aligned} \tag{53}$$

which is a constant, independent from the shape of the potential. If we divide Eq. (53) by the characteristic thermal length $\sqrt{K_B T/m}/\gamma$, we get

$$\lim_{t\to\infty} \sqrt{4\pi\gamma t}\left\langle -\frac{dV}{dx}\right\rangle = \gamma\sqrt{k_B T m}, \tag{54}$$

which has the physical units of force. For further details concerning the infinite ergodic concept we refer to Refs. [36–38].

V. INFINITE ERGODICITY IN GEOMETRIC BROWNIAN MOTION

By invoking the infinite ergodicity concept discussed in the previous section, we now try to give significance to the nonnormalized asymptotic solutions for the case $\alpha = 1/2$ (Fisk-Stratonovich interpretation) in the equation

$$\frac{dx}{dt} = -H_0 x^n + G_0 x \xi(t). \tag{55}$$

We still consider the relationship $G_0^2 \frac{n-1}{H_0} > 0$ to be valid through the hypotheses $n-1 < 0$ and $H_0 < 0$. These assumptions prevent us from having blow-up or explosion phenomena, as discussed in Appendix B. Hence, from Eq. (32), the nonnormalized asymptotic density takes the form

$$W_{as}(x) \sim \frac{1}{x}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right). \tag{56}$$

We remark that the exponential term approaches 1 for $x \to \infty$ because of the assumptions $n-1 < 0$ and $H_0 < 0$, exactly like the Boltzmann exponential of the previous section in the nonconfining regions. From the previously developed theory of geometric Brownian motion, we know that without drift we have the exact solution of the Fokker-Planck equation given

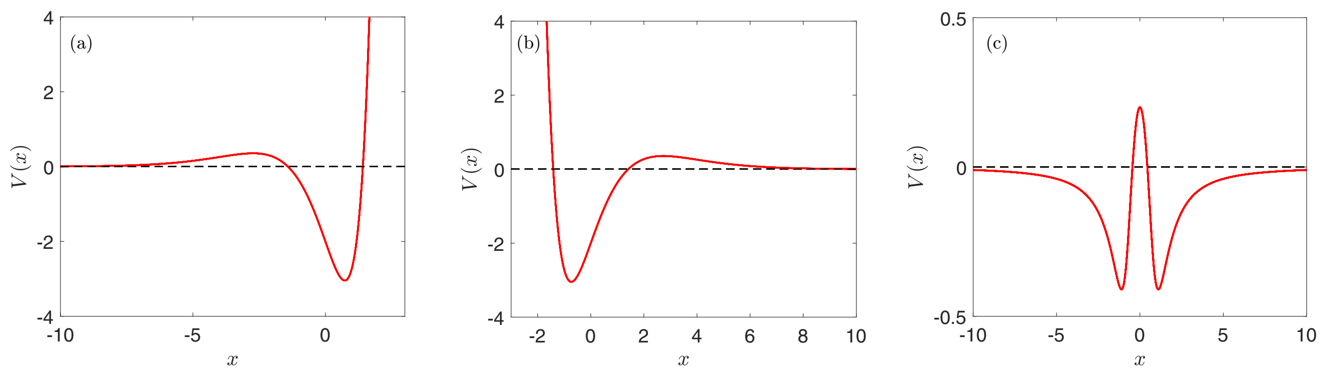

FIG. 4. Three examples of potential energy involving infinite ergodicity: (a) nonconfining on the left, $\lim_{x\to-\infty} V(x) = 0$ and $\lim_{x\to+\infty} V(x) = +\infty$; (b) nonconfining on the right, $\lim_{x\to-\infty} V(x) = +\infty$ and $\lim_{x\to+\infty} V(x) = 0$; (c) nonconfining on both the left and the right, $\lim_{x\to-\infty} V(x) = 0$ and $\lim_{x\to+\infty} V(x) = 0$.

by Eq. (22) (with $H_0 = 0$). In fact, with $\alpha = 1/2$ and $H_0 = 0$ we get

$$W(x,t) = \frac{\exp\left[-\frac{(\log\frac{x}{\mu_0})^2}{4G_0^2 t}\right]}{2xG_0\sqrt{\pi t}}, \tag{57}$$

corresponding to the initial condition $W(x,0) = \delta(x-\mu_0)$. In analogy with the treatment of the overdamped Langevin equation in the previous section, here we can assume a solution for long times of the process with $H_0 \neq 0$ and $\alpha = 1/2$ as a combination of Eqs. (56) and (57). We therefore have for long times

$$W(x,t) \underset{t\to\infty}{\sim} \frac{\exp\left[-\frac{(\log\frac{x}{\mu_0})^2}{4G_0^2 t}\right]}{2xG_0\sqrt{\pi t}} \exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right), \tag{58}$$

or, equivalently,

$$W(x,t) \underset{t\to\infty}{\sim} \frac{1}{2xG_0\sqrt{\pi t}} \exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right), \tag{59}$$

where we have used the limiting property $\lim_{t\to\infty} \exp[-(\log\frac{x}{\mu_0})^2/(4G_0^2 t)] = 1$. To verify this conjecture we have to establish that Eq. (59) actually is the solution for long times of the following Fokker-Planck equation

$$\frac{\partial W}{\partial t} = H_0 \frac{\partial}{\partial x}(x^n W) + G_0^2 \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}(xW)\right]. \tag{60}$$

By substituting Eq. (59) into Eq. (60), we see that all the terms behaving as $t^{-1/2}$ (the leading terms) cancel each other out and only one negligible term remains of order $t^{-3/2}$. This term is again negligible as it tends to zero much faster than the others and therefore is not relevant for long times. Now, from Eq. (59) we obtain the important expression

$$\lim_{t\to\infty} 2G_0\sqrt{\pi t}\,W(x,t) = \frac{1}{x}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right). \tag{61}$$

Here, the right hand side is the so-called invariant density (see Fig. 5). Also, in this case we can define an arbitrary observable $\mathcal{O}(x)$ and introduce its expectation value (as an ensemble average) as

$$\langle\mathcal{O}(x)\rangle(t) = \int_0^{+\infty} \mathcal{O}(x)W(x,t)dx, \tag{62}$$

where we considered the integration interval $(0, +\infty)$ to be consistent with the geometric Brownian motion phase space. Asymptotically, we get

$$\lim_{t\to\infty} 2G_0\sqrt{\pi t}\langle\mathcal{O}(x)\rangle(t) = \int_0^{+\infty} \frac{\mathcal{O}(x)}{x}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right)dx. \tag{63}$$

We now give an application of Eq. (63) with an observable defined as the power $\mathcal{O}(x) = x^s$ (where s is a real number). For this we simply rewrite the infinite ergodicity expression as

$$\lim_{t\to\infty} 2G_0\sqrt{\pi t}\langle x^s\rangle(t) = \int_0^{+\infty} x^{s-1}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-1}}{n-1}\right)dx, \tag{64}$$

where the integral converges if $s/(n-1) > 0$, provided that $G_0^2(n-1)/H_0 > 0$. This is true since the integral has the same form discussed in Eq. (33). So, we have the closed-form

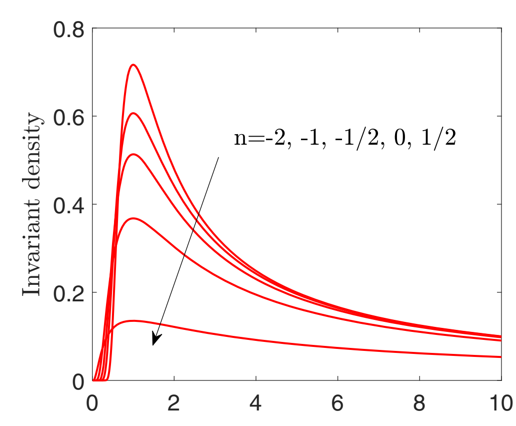

FIG. 5. Invariant density defined in Eq. (61) for the geometric Brownian motion with nonlinear drift. We used the parameters $H_0 = -1$, $G_0 = 1$ and a variable exponent n.

expression

$$\lim_{t\to\infty} 2G_0\sqrt{\pi t}\langle x^s\rangle(t) = \frac{1}{|n-1|}\left(\frac{n-1}{H_0}G_0^2\right)^{\frac{s}{n-1}}\Gamma\left(\frac{s}{n-1}\right) \quad (65)$$

coming from Eq. (40). An interesting special case corresponds to $s = n - 1$ and yields

$$\lim_{t\to\infty} 2G_0\sqrt{\pi t}\langle x^{n-1}\rangle(t) = \frac{\mathrm{sgn}(n-1)}{H_0}G_0^2 = \frac{1}{|H_0|}G_0^2, \quad (66)$$

where sgn(z) is the signum function extracting the sign of the real number z, and where we considered that $n - 1 < 0$ and $H_0 < 0$. We note that the result in Eq. (66) is independent of n, i.e., independent of the shape of the forcing term in Eq. (55). This result can be put in analogy with the asymptotic property of the average value of the force obtained in Eq. (54). Indeed, if we rewrite the stochastic differential equation in Eq. (55) in a form to render the space-dependent friction term on the left-hand side explicit, we have

$$\frac{1}{x}\frac{dx}{dt} = -H_0x^{n-1} + G_0\xi(t). \quad (67)$$

We can identify the observable x^{n-1} exactly as the force acting on the system.

VI. A FURTHER GENERALIZATION

We finally consider the generalization of the geometric Brownian motion given in Eq. (31) where the multiplicative noise term is now given by a nonlinear power with exponent m, as stated in the introduction in Eq. (4),

$$\frac{dx}{dt} = -H_0x^n + G_0x^m\xi(t). \quad (68)$$

The stochastic force-part of the equation, i.e., $dx/dt = G_0x^m\xi(t)$, was considered in detail already in the context of the infinite ergodicity concept in Ref. [38], using a slightly different notation. Several applications of this model to turbulence or ecosystems were mentioned in that paper. Our model represents a generalization of this model by combining it with a nonlinear drift term. The stochastic problem in Eq. (68) is associated with the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = H_0\frac{\partial}{\partial x}(x^nW) + G_0^2\frac{\partial}{\partial x}\left\{x^{2m\alpha}\frac{\partial}{\partial x}[x^{2m(1-\alpha)}W]\right\}. \quad (69)$$

For now, we first only assume that $m \neq 1$ in order to not reconsider the case already studied and search for an asymptotic solution $W_{as}(x)$ for the Fokker-Planck equation

$$0 = H_0x^nW_{as} + G_0^2x^{2m\alpha}\frac{\partial}{\partial x}[x^{2m(1-\alpha)}W_{as}]. \quad (70)$$

With the same technique used to solve Eq. (25), we find

$$W_{as}(x) = \frac{K}{x^{2m(1-\alpha)}}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right), \quad (71)$$

for which we have to require that $n - 2m + 1 \neq 0$. As before, W_{as} is normalizable with finite K when the integral over $(0, +\infty)$ is finite:

$$\frac{1}{K} = \int_0^{+\infty}\frac{1}{x^{2m(1-\alpha)}}\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)dx. \quad (72)$$

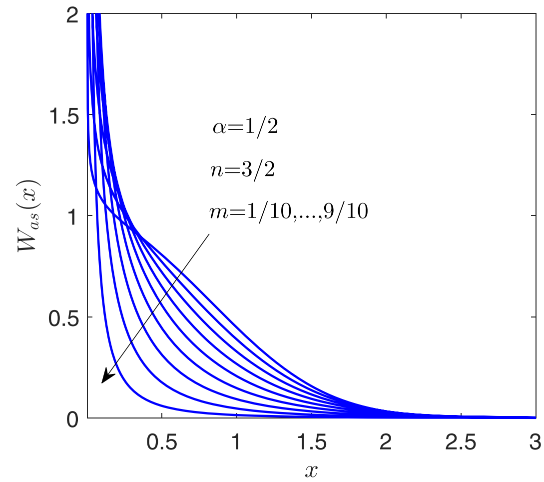

FIG. 6. Example of asymptotic distributions in the region defined by Eq. (73). We implemented Eq. (76) with the parameters $H_0 = 3/2$ and $G_0 = 1$.

The analysis follows that performed for Eq. (33), and generalizes it for finite $m \neq 1$.

(i) The term $x^{-2m(1-\alpha)}$ is convergent for $x \to 0$ when $2m(1-\alpha) < 1$. The exponential term ensures the convergence for $x \to \infty$ if $a > 0$ and $n - 2m + 1 > 0$. Finally, Eq. (72) is convergent if

$$2m(1-\alpha) < 1,\ \ H_0 > 0,\ \text{and}\ n - 2m + 1 > 0; \quad (73)$$

(ii) The term $x^{-2m(1-\alpha)}$ ensures the convergence for $x \to \infty$ if $2m(1-\alpha) > 1$. The exponential term provides the convergence for $x \to 0$ if $a < 0$ and $n - 2m + 1 < 0$. Hence, Eq. (72) is convergent also if

$$2m(1-\alpha) > 1,\ \ H_0 < 0,\ \text{and}\ n - 2m + 1 < 0. \quad (74)$$

The calculation of the integral in Eq. (72) can be done by the same method used before, and we get

$$\frac{1}{K} = \frac{\Gamma\left(\frac{2m\alpha-2m+1}{n-2m+1}\right)}{|n-2m+1|}\left(\frac{n-2m+1}{H_0}G_0^2\right)^{\frac{2m\alpha-2m+1}{n-2m+1}}. \quad (75)$$

Therefore, the asymptotic density reads as

$$W_{as}(x) = \frac{|n-2m+1|\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{\left(\frac{n-2m+1}{H_0}G_0^2\right)^{\frac{2m\alpha-2m+1}{n-2m+1}}\Gamma\left(\frac{2m\alpha-2m+1}{n-2m+1}\right)x^{2m(1-\alpha)}}, \quad (76)$$

which is correct for

$$G_0^2\frac{n-2m+1}{H_0} > 0,\ \text{and}\ \frac{2m\alpha-2m+1}{n-2m+1} > 0, \quad (77)$$

coming form Eqs. (73) and (74), and generalizing Eq. (36). In Figs. 6 and 7 we show the shape of the asymptotic density for different values of the parameters in the two regions, defined by Eqs. (73) and (74), respectively. We see that in the first region we have a singularity for $x = 0$, whereas in the second region the density is regular everywhere. Moreover, we remark that in both regions identified by Eqs. (73) and

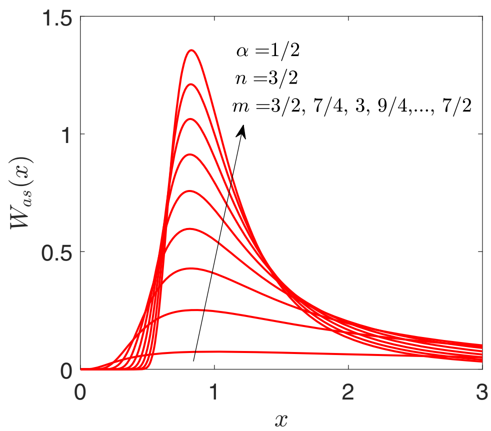

FIG. 7. Example of asymptotic distributions in the Itô-side region defined by Eq. (74). We implemented Eq. (76) with the parameters $H_0 = -3/2$ and $G_0 = 1$.

(74), the criterion for the absence of blow-up or explosions, proved in Appendix B, is perfectly fulfilled.

To obtain the infinite ergodic property for this system, we need to know the general solution of Eq. (68) without the forcing term, i.e., for $H_0 = 0$, for $m \neq 1$,

$$\frac{dx}{dt} = G_0 x^m \xi(t). \tag{78}$$

This problem has been investigated in detail in Ref. [37], and the solution has been found by showing a connection with the so-called Bessel process [53,54]. With our notation, we can say that the solution of Eq. (69) with $H_0 = 0$, rewritten here as

$$\frac{\partial W}{\partial t} = G_0^2 \frac{\partial}{\partial x}\left\{x^{2m\alpha}\frac{\partial}{\partial x}[x^{2m(1-\alpha)}W]\right\}, \tag{79}$$

is given by [37]

$$W(x,t) = \frac{\mu_0^{\frac{1}{2}(1-2m\alpha)} x^{\frac{1}{2}(1-4m+2m\alpha)}}{2G_0^2(1-m)t} \times \exp\left[-\frac{\mu_0^{2(1-m)} + x^{2(1-m)}}{4G_0^2(1-m)^2 t}\right] \times I_{\frac{1-2m\alpha}{2(m-1)}}\left(\frac{\mu_0^{1-m} x^{1-m}}{2G_0^2(1-m)^2 t}\right), \tag{80}$$

for $x \geqslant 0$ (with reflecting boundary condition at $x = 0$), and for the initial condition $W(x,0) = \delta(x-\mu_0)$. Here $I_\nu(z)$ is the modified Bessel function of the first kind (of order ν and argument z) [45–47]. Importantly, this solution is valid when $0 \leqslant m < 1$ and $2m\alpha - 2m + 1 > 0$, and represents a time evolution that does not have a stationary PDF. For $\alpha = 1/2$ (Fisk-Stratonovich interpretation), we can use the relation [45–47]

$$I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cosh(z), \tag{81}$$

and obtain the particular solution

$$W(x,t) = \frac{1}{G_0\sqrt{\pi t}x^m}\cosh\left(\frac{\mu_0^{1-m}x^{1-m}}{2G_0^2(1-m)^2 t}\right) \times \exp\left[-\frac{\mu_0^{2(1-m)} + x^{2(1-m)}}{4G_0^2(1-m)^2 t}\right]. \tag{82}$$

This expression can be rewritten as

$$W(x,t) = \frac{1}{2G_0\sqrt{\pi t}x^m}\exp\left[-\frac{\left(x^{1-m} - \mu_0^{1-m}\right)^2}{4G_0^2(1-m)^2 t}\right] + \frac{1}{2G_0\sqrt{\pi t}x^m}\exp\left[-\frac{\left(x^{1-m} + \mu_0^{1-m}\right)^2}{4G_0^2(1-m)^2 t}\right]. \tag{83}$$

From the point of view of the physical interpretation, this form shows the superposition of an incident density (the first line) generated by the initial condition at $x = \mu_0$, and a reflected density (the second line) generated by the reflecting boundary condition at $x = 0$. Moreover, if $m = 0$, incident and reflected densities are Gaussian functions, as to be expected with additive noise.

As always, the Stratonovich interpretation is closer to the physical understanding that can be attributed to the evolution of a stochastic system. In this case, the other interpretations with $\alpha \neq 1/2$ are able to break the symmetry between incident and reflected densities, as mathematically described by the Bessel function in Eq. (80). Both Eqs. (80) and (82) can be proved by direct substitution in Eq. (79). By means of these solutions, we can study the asymptotic behavior, for large values of t, of the equation $dx/dt = G_0 x^m \xi(t)$. To do this, we use the property [45–47]

$$I_\nu(z) \underset{z\to 0}{\sim} \left(\frac{1}{2}z\right)^\nu \frac{1}{\Gamma(\nu+1)}, \tag{84}$$

and from Eq. (80) we obtain

$$W(x,t) \underset{t\to\infty}{\sim} \frac{[2(1-m)]^{(1-2\alpha)\frac{m}{1-m}}}{\left(G_0^2 t\right)^{\frac{2m\alpha-2m+1}{2(1-m)}}\Gamma\left[\frac{2m\alpha-2m+1}{2(1-m)}\right]x^{2m(1-\alpha)}}. \tag{85}$$

In the particular case with $\alpha = 1/2$, we use the Gamma function value $\Gamma(1/2) = \sqrt{\pi}$ [45–47], and we obtain from Eq. (82) or Eq. (83) the simpler asymptotic behavior

$$W(x,t) \underset{t\to\infty}{\sim} \frac{1}{G_0\sqrt{\pi t}x^m}. \tag{86}$$

Summing up, on the one hand, we can say that the process with drift term, see Eq. (68), exhibits an equilibrium asymptotic solution when Eq. (73) or Eq. (74) is satisfied. On the other hand, for the equation without forcing term, see Eq. (78), there is no equilibrium and we know the asymptotic evolution when $0 \leqslant m < 1$ and $2m\alpha - 2m + 1 > 0$. The idea of the infinite ergodicity is to give meaning to the equilibrium solution $W_{as}(x)$ of Eq. (68) even when it cannot be normalized. Hence, we consider the conditions $0 \leqslant m < 1$ and $2m\alpha - 2m + 1 > 0$, under which we know the asymptotic solution of Eq. (78), and we add the assumptions $H_0 < 0$ and $n - 2m + 1 < 0$ in such a way that $W_{as}(x)$ it is not normalizable. These conditions also ensure the absence of blow-up phenomena, as discussed

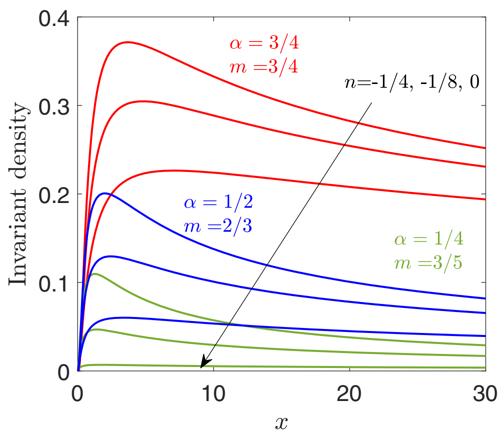

FIG. 8. Invariant density defined in Eqs. (89) and (90) for the generalized geometric Brownian motion with nonlinear drift. We used the parameters $H_0 = -1$, $G_0 = 1$, variable exponents n and m, and variable parameter α.

in Appendix B. When this set of conditions is satisfied, we can try to merge Eqs. (76) and (85) in order to get the asymptotic behavior. This is facilitated by the fact that in both formulas there is the same power $x^{2m(1-\alpha)}$ in the denominator. We therefore propose to consider

$$W(x,t) \underset{t\to\infty}{\sim} \frac{[2(1-m)]^{(1-2\alpha)\frac{m}{1-m}} \exp\left(-\frac{a}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{\left(G_0^2 t\right)^{\frac{2m\alpha-2m+1}{2(1-m)}} \Gamma\left[\frac{2m\alpha-2m+1}{2(1-m)}\right] x^{2m(1-\alpha)}}. \quad (87)$$

If $\alpha = 1/2$, we can merge Eqs. (76) and (86) and have

$$W(x,t) \underset{t\to\infty}{\sim} \frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{G_0\sqrt{\pi t}\, x^m}. \quad (88)$$

These proposals should represent the asymptotic behavior of Eq. (68) when $0 \leqslant m < 1$, $2m\alpha - 2m + 1 > 0$, $H_0 < 0$, and $n - 2m + 1 < 0$. The verification by direct substitution into the Fokker-Planck equation proceeds as before. So, Eq. (87) can be rewritten as

$$\lim_{t\to\infty} \frac{\Gamma\left[\frac{2m\alpha-2m+1}{2(1-m)}\right]\left(G_0^2 t\right)^{\frac{2m\alpha-2m+1}{2(1-m)}} W(x,t)}{[2(1-m)]^{(1-2\alpha)\frac{m}{1-m}}} = \frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{x^{2m(1-\alpha)}}, \quad (89)$$

and Eq. (88) for $\alpha = 1/2$ as

$$\lim_{t\to\infty} G_0\sqrt{\pi t}\, W(x,t) = \frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{x^m}. \quad (90)$$

These results represent the asymptotic evolution of the density $W(x,t)$, solving Eq. (69) for large values of t, in the case where there is no equilibrium PDF, i.e., they represent the generalization of Eq. (61), obtained previously for the case of geometric Brownian motion. In particular, Eq. (90) reduces to Eq. (61) when $m = 1$. The right hand sides of Eqs. (89) and (90) are therefore the so-called invariant densities of the system (see Fig. 8).

In an analogous fashion we can introduce an observable $\mathcal{O}(x)$ with its ensemble average defined in Eq. (62). From previous asymptotic results, we easily obtain

$$\lim_{t\to\infty} \frac{\Gamma\left[\frac{2m\alpha-2m+1}{2(1-m)}\right]\left(G_0^2 t\right)^{\frac{2m\alpha-2m+1}{2(1-m)}}}{[2(1-m)]^{(1-2\alpha)\frac{m}{1-m}}} \langle\mathcal{O}(x)\rangle(t) = \int_0^{+\infty} \frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{x^{2m(1-\alpha)}} \mathcal{O}(x)dx \quad (91)$$

for arbitrary values of α, and

$$\lim_{t\to\infty} G_0\sqrt{\pi t}\langle\mathcal{O}(x)\rangle(t) = \int_0^{+\infty} \frac{\exp\left(-\frac{H_0}{G_0^2}\frac{x^{n-2m+1}}{n-2m+1}\right)}{x^m} \mathcal{O}(x)dx \quad (92)$$

for $\alpha = 1/2$. Both results are valid when $0 \leqslant m < 1$, $2m\alpha - 2m + 1 > 0$, $H_0 < 0$, and $n - 2m + 1 < 0$. These two results again give important significance to asymptotic distributions even when the latter cannot be normalized and therefore represent a further example of infinite ergodic theory. They are valid only when the form of the observable $\mathcal{O}(x)$ renders the integral appearing in Eqs. (91) and (92) convergent.

VII. CONCLUSIONS

We have studied the stochastic process of geometric Brownian motion and some of its generalizations as given by Eq. (68) for general values of the drift exponent n, the diffusion exponent m, and the discretization parameter $0 \leqslant \alpha \leqslant 1$. The corresponding Fokker-Planck equation is readily written down, following established procedures. The study of the asymptotic probability distributions of the Fokker-Planck equation reveals that the normalizability of the PDF at large times is tied to general conditions on the exponents m, n, and α. We establish the conditions on the exponents n, m and on the discretization parameter α for which this is the case. Our—surprising—main finding for the case of the standard geometric Brownian noise with $m = 1$ is that the presence of a drift term in the stochastic equation allows to produce normalizable stationary PDFs provided $\alpha \neq 1/2$. If $\alpha = 1/2$ (Fisk-Stratonovich case), the concept of infinite ergodicity allows to derive a well-defined invariant density, defined on the right hand side of Eq. (61). In the generalizations for $m \neq 1$, our results link to the findings by Barkai and collaborators, notably those of Ref. [37]. In this case, we are able to find an invariant density for $\alpha = 1/2$ (Fisk-Stratonovich case), see the right hand side of Eq. (90), but also another invariant density for an arbitrary stochastic interpretation, see Eq. (89). In conclusion, we can say that infinite ergodic theory provides interesting results not only for classical statistical mechanics with additive noise, but also for more complex stochastic processes with multiplicative noise such as geometric Brownian motion or its generalizations. More specifically, the obtained results allows us to exactly determine the asymptotic behavior of physical observables in complex drift-diffusion driven systems even though we cannot find the general solution of the associated Fokker-Planck equation.

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APPENDIX A: PROBABILITY DENSITY FOR TIME-VARYING GEOMETRIC BROWNIAN MOTION

We prove here that the generalized log-normal distribution, given in Eq. (21), is the solution of the Fokker-Planck equation stated in Eq. (7) or Eq. (8). To this aim, we first calculate the partial derivative of the density with respect to time. Straightforward calculation leads to

$$\frac{\partial W(x,t)}{\partial t} = \frac{W(x,t)}{\sigma^4}\{G^2(\mu^2-\sigma^2)+G^2\log^2 x - 2\mu G^2 \log x - [H+(2\alpha-1)G^2](\mu-\log x)\sigma^2\}, \tag{A1}$$

where we have used Eqs. (19) and (20) for the values of μ and σ^2. In order to use the Fokker-Planck equation in the form of Eq. (7), we also need the following partial derivative

$$\frac{\partial[xW(x,t)]}{\partial x} = \frac{W(x,t)}{\sigma^2}(\mu-\log x). \tag{A2}$$

We now need to calculate $\partial^2[x^2 W(x,t)]/\partial x^2$ and we proceed in the following way:

$$\frac{\partial^2[x^2 W(x,t)]}{\partial x^2} = \frac{\partial[xW(x,t)]}{\partial x} + \frac{\partial}{\partial x}\left[x\frac{\partial[xW(x,t)]}{\partial x}\right]. \tag{A3}$$

The last term of Eq. (A3) can be developed as

$$\begin{aligned}\frac{\partial}{\partial x}\left[x\frac{\partial[xW(x,t)]}{\partial x}\right] &= \frac{\partial}{\partial x}\left[\frac{xW(x,t)}{\sigma^2}(\mu-\log x)\right]\\ &= \frac{\mu W}{\sigma^4}(\mu-\log x) - \frac{1}{\sigma^2}\left[\frac{W}{\sigma^2}(\mu-\log x)\log x + W\right].\end{aligned} \tag{A4}$$

We can now combine Eqs. (A3) and (A4) and we eventually get

$$\frac{\partial^2[x^2 W(x,t)]}{\partial x^2} = \frac{W(x,t)}{\sigma^4}(\mu^2-\sigma^2+\mu\sigma^2 + \log^2 x - \sigma^2\log x - 2\mu\log x). \tag{A5}$$

By means of Eqs. (A2) and (A5), we can then obtain the right hand side of the Fokker-Planck equation in Eq. (7)

$$\begin{aligned}&-(H+2\alpha G^2)\frac{\partial}{\partial x}(xW) + G^2\frac{\partial^2}{\partial x^2}(x^2 W)\\ &= \frac{W(x,t)}{\sigma^4}\{G^2(\mu^2-\sigma^2)+G^2\log^2 x - 2\mu G^2\log x - [H+(2\alpha-1)G^2](\mu-\log x)\sigma^2\},\end{aligned} \tag{A6}$$

which perfectly corresponds to the left hand side calculated in Eq. (A1). This finally proves the log-normal distribution for the time-varying geometric Brownian motion.

APPENDIX B: ON BLOW-UP OR EXPLOSION PHENOMENA

Blow-up or explosion phenomena are observed when solutions of a differential equation (ordinary or stochastic) tend to infinity as time approaches a finite value (explosion time). As a simple example, we can consider the ordinary differential equation $\frac{dx}{dt} = -H_0 x^n$, with n real, $x(0) = x_0 > 0$. Its integration, through the method of separation of variables, provides

$$x(t) = \left[x_0^{1-n} + (n-1)H_0 t\right]^{\frac{1}{1-n}}. \tag{B1}$$

We see that the solution reaches infinity in finite time if $n > 1$ and $H_0 < 0$. Then, we cannot have blow-up phenomena if $H_0 > 0$ (stability), or if $n < 1$ (subexponential growth). More in general, it is possible to prove that given $\frac{dx}{dt} = h(x)$, $x(0) = x_0$, $h(x) > 0$ for $x \geqslant x_0$, then $x(t)$ blows-up at time T if and only if $\int_{x_0}^{+\infty} h(x)^{-1}dx$ converges to T [55,56]. When this integral is divergent, we cannot observe explosion phenomena (Osgood criterion) [55].

In the case of a one-dimensional stochastic differential equation of the form $\frac{dx}{dt} = h(x) + g(x)\xi(t)$, with $x(0) = x_0 \in (\ell, r)$ the so-called Feller test for explosions solves the problem [57–59]. In this case, the explosion time is defined as $T = \inf\{t \geqslant 0 : x(t) \notin (\ell, r)\}$. In general, introducing the Feller function

$$v(x) = \int_{x_0}^{x}\int_{x_0}^{y}\exp\left(-2\int_z^y \frac{h(u)}{g^2(u)}du\right)\frac{dz}{g^2(z)}dy, \tag{B2}$$

we have that the probability $\Pr\{T = \infty\} = 1$ if and only if $v(\ell^+) = v(r^-) = \infty$ (i.e., the corresponding integrals diverge). The Feller test for explosions gives a precise description of the blow-up phenomena in finite time in terms of $h(x)$, $g(x)$, and x_0. It has been pointed out that the Feller test is equivalent to the Osgood criterion for $g(x) = constant$ [60]. Applying the Feller test to our equation

$$\frac{dx}{dt} = -H_0 x^n + G_0 x^m \xi(t), \tag{B3}$$

we obtain that we cannot have blow-up phenomena for $H_0 > 0$. In addition, when $H_0 < 0$, we have explosions in finite time if and only if $n > 2m - 1$ and $n > 1$ (or, equivalently, if and only if $n > \max\{2m-1, 1\}$) [61,62]. The same result holds on if $h(x)$ and $g(x)$ behave like powers at infinity (i.e., $h(x) \sim x^n$, $g(x) \sim x^m$ as $x \to \infty$).

It is important to underline that the Feller test and our specific application are valid within the Itô interpretation of the stochastic calculus. We prove here that the test for explosions for Eq. (B3) is independent of the stochastic interpretation. If we have a stochastic equation $\frac{dx}{dt} = h_\alpha(x) + g_\alpha(x)\xi(t)$, interpreted through the stochastic calculus with arbitrary parameter α, we have the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left[\left(h_\alpha + 2\alpha g_\alpha\frac{\partial g_\alpha}{\partial x}\right)W\right] + \frac{\partial^2}{\partial x^2}\left(g_\alpha^2 W\right). \tag{B4}$$

We can consider a different stochastic equation $\frac{dx}{dt} = h_0(x) + g_0(x)\xi(t)$, interpreted through the Itô calculus, and we get

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}(h_0 W) + \frac{\partial^2}{\partial x^2}\left(g_0^2 W\right). \tag{B5}$$

The two problems are formally equivalent if the two Fokker-Planck equations coincide, i.e., when

$$g_0(x) = g_\alpha(x), \tag{B6}$$

$$h_0(x) = h_\alpha(x) + 2\alpha g_\alpha(x)\frac{\partial g_\alpha(x)}{\partial x}. \tag{B7}$$

These rules allow the transition from an arbitrary stochastic interpretation to an Itô interpretation (the term added to h_α to get h_0 is sometimes called Wong-Zakai correction, as described in Chapter 3 of Ref. [63]). By using this result, we can state that the equation $\frac{dx}{dt} = -H_0x^n + G_0x^m\xi(t)$, within the arbitrary stochastic calculus with parameter α, is equivalent to the equation

$$\frac{dx}{dt} = -H_0x^n + 2m\alpha G_0^2x^{2m-1} + G_0x^m\xi(t), \tag{B8}$$

interpreted through the Itô calculus. We can discuss two cases: (i) if $n > 2m-1$, the term $-H_0x^n$ is dominant over $2m\alpha G_0^2x^{2m-1}$ and we have blow-up phenomena if and only if $H_0 < 0$, $n > 2m-1$, and $n > 1$; (ii) if $n < 2m-1$, the term $2m\alpha G_0^2x^{2m-1}$ is dominant over $-H_0x^n$ and therefore the condition to have explosions is $2m-1 > \max\{2m-1, 1\}$, which can never be satisfied. To conclude, Eq. (B3) exhibits blow-up phenomena if and only if $H_0 < 0$, $n > 2m-1$, and $n > 1$, regardless of the stochastic interpretation adopted. This result precludes explosion problems in all applications shown in this article.

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