Continuum mechanics and nonlinear elasticity

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1. Symbols

List of the most important tensor quantities used in the following sections

$\hat{F}$	deformation gradient
$\hat{G}$	inverse deformation gradient
$\hat{L}$	velocity gradient
J	deformation Jacobian
$\hat{B}$ and $\hat{C}$	left and right Cauchy tensors
$\hat{U}$ and $\hat{V}$	left and right stretching tensors
$\hat{R}$	rotation tensor
$\hat{\eta}$	Green-Lagrange tensor
$\hat{e}$	Almansi-Eulero tensor
$\hat{J}_L$	Lagrangian displacement gradient
$\hat{J}_E$	Eulerian displacement gradient
$\hat{D}$	rate of deformation tensor
$\hat{W}$	spin tensor
$\hat{T}$	Cauchy stress tensor
$\hat{T}^{1\mathcal{PK}}$	first Piola-Kirchhoff stress tensor
$\hat{T}^{2\mathcal{PK}}$	second Piola-Kirchhoff stress tensor
$\hat{J}$	small-strain displacement gradient
$\hat{\epsilon}$	small-strain tensor
$\hat{\Omega}$	local rotation tensor
$\hat{\mathcal{C}}$	stiffness tensor

2. Lagrangian versus Eulerian formalism

The motion of a body is typically referred to a reference configuration $\Omega_0 \subset \Re^3$, which is often chosen to be the undeformed configuration. After the deformation the body occupies the current configuration $\Omega_t \subset \Re^3$. Thus, the current coordinates ($\vec{x} \in \Omega_t$) are expressed in terms of the reference coordinates ($\vec{X} \in \Omega_0$):

$$\vec{X} \mapsto \vec{x} = \mathcal{F}_t\left(\vec{X}\right) \tag{2.1}$$

where $\mathcal{F}_t$ is the transformation function at any time t (see Fig. 1). More explicitely, it means that

$$\begin{aligned} x_1 &= x_1\left(X_1, X_2, X_3, t\right) \\ x_2 &= x_2\left(X_1, X_2, X_3, t\right) \\ x_3 &= x_3\left(X_1, X_2, X_3, t\right) \end{aligned} \tag{2.2}$$

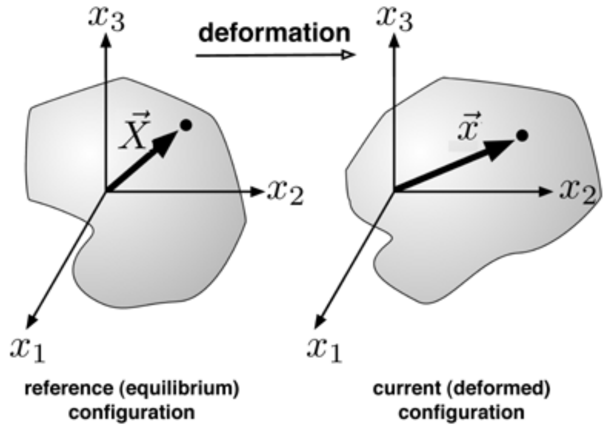

Figure 1. Reference configuration and current configuration after a deformation.

We call the set ($\vec{X}$ and t) Lagrangian coordinates, named after Joseph Louis Lagrange [1736-1813], or material coordinates, or reference coordinates. The application of these coordinates is called Lagrangian description or reference description. We can obtain also the inverse function of Eq.(2.1) in the form

$$\vec{x} \mapsto \vec{X} = \mathcal{F}_t^{-1}(\vec{x}) \tag{2.3}$$

or, in components

$$\begin{aligned} X_1 &= X_1(x_1, x_2, x_3, t) \\ X_2 &= X_2(x_1, x_2, x_3, t) \\ X_3 &= X_3(x_1, x_2, x_3, t) \end{aligned} \tag{2.4}$$

The set ($\vec{x}$ and t) is called Eulerian coordinates, named after Leonhard Euler [1707-1783], or space coordinates, and their application is said Eulerian description or spatial description. The Lagrangian coordinates were introduced by Euler in 1762, while Jean le Rond D'Alembert [1717-1783] was the first to use the Eulerian coordinates in 1752. In general Continuum Mechanics Lagrangian coordinates and the reference description are the most common. The same holds true in solid Mechanics. However, in Fluid Mechanics, due to large displacements and complex deformations, it is usually necessary and most practical to use Eulerian coordinates and spatial description.

One of the key quantities in deformation analysis is the deformation gradient of Ω_t relative to the reference configuration Ω_0, denoted $\hat{F}$, which gives the relationship of a material line $d\vec{X}$ before deformation to the line $d\vec{x}$ (consisting of the same material as $d\vec{X}$) after deformation. It is defined as

$$\vec{x} = \mathcal{F}_t\left(\vec{X}\right) : \hat{F}\left(\vec{X}, t\right) = \vec{\nabla}_{\vec{X}} \mathcal{F}_t\left(\vec{X}\right) \Rightarrow d\vec{x} = \hat{F}\left(\vec{X}, t\right) d\vec{X} \tag{2.5}$$

Its components are given by

$$F_{iK} = \frac{\partial x_i}{\partial X_K} \;\forall (i, K) \in \{1, 2, 3\}^2 \tag{2.6}$$

As before, we can define a deformation gradient $\hat{G}$ of the inverse function relating Ω_0 to the current configuration Ω_t

$$\vec{X} = \mathcal{F}_t^{-1}\left(\vec{x}\right): \ \hat{G}\left(\vec{x}, t\right) = \vec{\nabla}_{\vec{x}}\mathcal{F}_t^{-1}\left(\vec{x}\right) \Rightarrow d\vec{X} = \hat{G}\left(\vec{x}, t\right) d\vec{x} \tag{2.7}$$

In components, it assumes the form

$$G_{Ki} = \frac{\partial X_k}{\partial x_i} \ \forall (i, K) \in \{1, 2, 3\}^2 \tag{2.8}$$

Of course, the tensors $\hat{F}$ and $\hat{G}$ are related by the relationships

$$\hat{G}\left(\mathcal{F}_t\left(\vec{X}\right), t\right) = \hat{F}^{-1}\left(\vec{X}, t\right) \tag{2.9}$$

$$\hat{F}\left(\mathcal{F}_t^{-1}\left(\vec{x}\right), t\right) = \hat{G}^{-1}\left(\vec{x}, t\right) \tag{2.10}$$

In fact, $\hat{F}$ is a Lagrangian tensor while $\hat{G}$ is an Eulerian tensor. The velocity and acceleration fields, related to the trajectory of the particle starting at $\vec{X}$ (Lagrangian description) are given by

$$\vec{v}\left(\vec{X}, t\right) = \frac{\partial \vec{x}}{\partial t}\left(\vec{X}, t\right) \tag{2.11}$$

$$\vec{a}\left(\vec{X}, t\right) = \frac{\partial^2 \vec{x}}{\partial t^2}\left(\vec{X}, t\right) \tag{2.12}$$

On the other hand, the velocity and acceleration fields in the Euler description are given by

$$\vec{v}\left(\vec{x}, t\right) = \frac{\partial \vec{x}}{\partial t}\left(\mathcal{F}_t^{-1}\left(\vec{x}\right), t\right) \tag{2.13}$$

$$\vec{a}\left(\vec{x}, t\right) = \frac{\partial^2 \vec{x}}{\partial t^2}\left(\mathcal{F}_t^{-1}\left(\vec{x}\right), t\right) \tag{2.14}$$

Any time-dependent scalar, vector, or tensor field can be regarded as a function of $(\vec{X},\ t)$ (Lagrangian or material variables) or $(\vec{x},\ t)$ (Eulerian or spatial variables) whenever the motion $\vec{x} = \mathcal{F}_t\left(\vec{X}\right)$ is given. For example, for a scalar field we can write $\phi\left(\vec{x}, t\right) = \Phi\left(\vec{X}, t\right)$ where

$$\Phi\left(\vec{X}, t\right) = \phi\left(\mathcal{F}_t\left(\vec{X}\right), t\right) \tag{2.15}$$

The time derivative of the field $\Phi\left(\vec{X}, t\right)$ can be calculated as

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \vec{x}} \cdot \vec{v} \tag{2.16}$$

Instead of using different symbols for the quantities (i.e. ϕ and Φ) in the Lagrangian and Eulerian descriptions, we can use the dot for the Lagrangian or material derivative ($\dot{\phi}$) and the partial differentiation symbol ($\frac{\partial \phi}{\partial t}$) for the Eulerian or spatial derivative: therefore, Eq.(2.16) assumes the simpler form

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \vec{x}} \cdot \vec{v} \tag{2.17}$$

The Eulerian tensor

$$\hat{L} = \frac{\partial \vec{v}}{\partial \vec{x}} \tag{2.18}$$

with components

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{2.19}$$

satisfies the important relation

$$\dot{\hat{F}} = \hat{L}\hat{F} \tag{2.20}$$

It can be proved as follows

$$\dot{\hat{F}} = \frac{\partial}{\partial t}\frac{\partial}{\partial \vec{X}}\mathcal{F}_t\left(\vec{X}\right) = \frac{\partial}{\partial \vec{X}}\frac{\partial}{\partial t}\mathcal{F}_t\left(\vec{X}\right) = \frac{\partial \vec{v}}{\partial \vec{X}} = \frac{\partial \vec{v}}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial \vec{X}} = \hat{L}\hat{F} \tag{2.21}$$

It is also important an inverse relation given by

$$\dot{\hat{F}}^{-1} = -\hat{G}\hat{L} \tag{2.22}$$

Since $\frac{d}{dt}\left(\hat{F}^{-1}\hat{F}\right) = 0$ we have $\dot{\hat{F}}^{-1} = -\hat{F}^{-1}\dot{\hat{F}}\hat{F}^{-1}$ (where $\dot{\hat{F}}^{-1}$ represents the Lagrangian time derivative of the inverse of $\hat{F}$) and, therefore, we obtain the proof of Eq.(2.22)

$$\dot{\hat{F}}^{-1} = -\hat{F}^{-1}\dot{\hat{F}}\hat{F}^{-1} = -\hat{F}^{-1}\hat{L}\hat{F}\hat{F}^{-1} = -\hat{F}^{-1}\hat{L} = -\hat{G}\hat{L} \tag{2.23}$$

2.1. Derivative of a volume integral

We consider a subset $\mathcal{P}_t \subset \Omega_t$ which is the time deformed version of $\mathcal{P}_0 \subset \Omega_0$. We search a property giving the time derivative of an arbitrary volume integral. In this context, the symbol d/dt can be used when it is applied to a quantity depending only on the time t. In fact, in this case, there is no ambuguity. As before we consider a scalar field ϕ and, through a change of variables between Eulerian and Lagrangian coordinates, we obtain

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi d\vec{x} = \frac{d}{dt}\int_{\mathcal{P}_0}\phi J d\vec{X} \tag{2.24}$$

where J is the determinant of the deformation gradient

$$J = \det\frac{\partial \vec{x}}{\partial \vec{X}} = \det\hat{F} \tag{2.25}$$

Then, the time derivation can enter the integral written in the reference configuration

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi d\vec{x} = \int_{\mathcal{P}_0}\frac{d}{dt}(\phi J)\, d\vec{X} = \int_{\mathcal{P}_0}\left(\dot{\phi}J + \phi\dot{J}\right) d\vec{X} \tag{2.26}$$

The derivative of a determinant follows the rule

$$\frac{d}{dt}\det\hat{F} = \left(\det\hat{F}\right)\mathrm{tr}\left(\dot{\hat{F}}\hat{F}^{-1}\right) \tag{2.27}$$

From Eq.(2.20) we obtain $\dot{\hat{F}}\hat{F}^{-1} = \hat{L}$ and, therefore,we have

$$\dot{J} = J\mathrm{tr}\left(\hat{L}\right) = J\vec{\nabla}_{\vec{x}}\cdot\vec{v} \tag{2.28}$$

So

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi d\vec{x}=\int_{\mathcal{P}_0}\left(\dot{\phi}+\phi\vec{\nabla}_{\vec{x}}\cdot\vec{v}\right)Jd\vec{X}=\int_{\mathcal{P}_t}\left(\dot{\phi}+\phi\vec{\nabla}_{\vec{x}}\cdot\vec{v}\right)d\vec{x} \tag{2.29}$$

Since $\dot{\phi}=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial\vec{x}}\cdot\vec{v}$ we obtain

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi d\vec{x}=\int_{\mathcal{P}_t}\left(\frac{\partial\phi}{\partial t}+\vec{\nabla}_{\vec{x}}\phi\cdot\vec{v}+\phi\vec{\nabla}_{\vec{x}}\cdot\vec{v}\right)d\vec{x} \tag{2.30}$$

or, finally

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi d\vec{x}=\int_{\mathcal{P}_t}\left[\frac{\partial\phi}{\partial t}+\vec{\nabla}_{\vec{x}}\cdot(\phi\vec{v})\right]d\vec{x} \tag{2.31}$$

This property has been called Reynolds theorem or transport theorem. It is the most important result used to obtain the balance equations for continuum materials. If $\phi=1$ we obtain

$$\frac{d}{dt}\int_{\mathcal{P}_t}d\vec{x}=\int_{\mathcal{P}_t}\vec{\nabla}_{\vec{x}}\cdot\vec{v}d\vec{x} \tag{2.32}$$

which represent the rate of variation of the volume of the region $\mathcal{P}_t$.

2.2. Derivative of a surface integral

We begin by describing the deformation of a given surface moving from the reference to the current configuration. We therefore consider a surface $\vec{X}=\vec{X}(\alpha,\beta)$ in the reference configuration described in parametric form by two parameters α and β. The deformed surface in the current configuration is given by $\vec{x}=\mathcal{F}_t\left(\vec{X}(\alpha,\beta)\right)$. We define $\vec{N}dS$ and $\vec{n}ds$ as the unit normal vector multiplied by the area element in the reference and in the current configuration, respectively. From standard differential geometry we have

$$\vec{N}dS=\frac{\partial\vec{X}}{\partial\alpha}\wedge\frac{\partial\vec{X}}{\partial\beta}d\alpha d\beta \tag{2.33}$$

The deformed version can be straightforwardly obtained as

$$\begin{aligned}\vec{n}ds&=\frac{\partial\vec{x}}{\partial\alpha}\wedge\frac{\partial\vec{x}}{\partial\beta}d\alpha d\beta=\left(\frac{\partial\vec{x}}{\partial\vec{X}}\frac{\partial\vec{X}}{\partial\alpha}\right)\wedge\left(\frac{\partial\vec{x}}{\partial\vec{X}}\frac{\partial\vec{X}}{\partial\beta}\right)d\alpha d\beta\\&=\left(\hat{F}\frac{\partial\vec{X}}{\partial\alpha}\right)\wedge\left(\hat{F}\frac{\partial\vec{X}}{\partial\beta}\right)d\alpha d\beta\end{aligned} \tag{2.34}$$

The last expression can be written component by component

$$n_i ds=\varepsilon_{ijk}F_{js}\frac{\partial X_s}{\partial\alpha}F_{kt}\frac{\partial X_t}{\partial\beta}d\alpha d\beta$$

and it can be multiplied by F_{ir} on both sides

$$F_{ir}n_i ds=\varepsilon_{ijk}F_{ir}F_{js}F_{kt}\frac{\partial X_s}{\partial\alpha}\frac{\partial X_t}{\partial\beta}d\alpha d\beta$$

Since $\varepsilon_{ijk}F_{ir}F_{js}F_{kt}=\det\hat{F}\varepsilon_{rst}$ we obtain

$$F_{ir}n_i ds=J\varepsilon_{rst}\frac{\partial X_s}{\partial\alpha}\frac{\partial X_t}{\partial\beta}d\alpha d\beta$$

or

$$\hat{F}^T \vec{n} ds = J \frac{\partial \vec{X}}{\partial \alpha} \wedge \frac{\partial \vec{X}}{\partial \beta} d\alpha d\beta = J \vec{N} dS \tag{2.35}$$

and finally we have obtained the relationship between $\vec{N}dS$ and $\vec{n}ds$

$$\vec{n} ds = J \hat{F}^{-T} \vec{N} dS \tag{2.36}$$

This property has been called Nanson theorem. Now, it is interesting to evaluate the time derivative of the surface integral of a vector field $\vec{a}$. It can be brought back to the reference configuration as

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{S}_t} \vec{a} \cdot \vec{n} ds &= \frac{d}{dt} \int_{\mathcal{S}_0} \vec{a} \cdot J \hat{F}^{-T} \vec{N} dS \\ &= \int_{\mathcal{S}_0} \left[\dot{\vec{a}} \cdot J \hat{F}^{-T} + \vec{a} \cdot \dot{J} \hat{F}^{-T} + \vec{a} \cdot J \dot{\hat{F}}^{-T} \right] \vec{N} dS \end{aligned} \tag{2.37}$$

Now $\dot{J} = J \mathrm{tr}\left(\hat{L}\right) = J \vec{\nabla}_{\vec{x}} \cdot \vec{v}$ and $\dot{\hat{F}}^{-T} = -\hat{F}^{-T} \dot{\hat{F}}^{T} \hat{F}^{-T}$ and therefore

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{S}_t} \vec{a} \cdot \vec{n} ds &= \int_{\mathcal{S}_0} \left[\dot{\vec{a}} \cdot J \hat{F}^{-T} + \vec{a} \cdot J \vec{\nabla}_{\vec{x}} \cdot \vec{v} \hat{F}^{-T} - \vec{a} \cdot J \hat{F}^{-T} \dot{\hat{F}}^{T} \hat{F}^{-T} \right] \vec{N} dS \\ &= \int_{\mathcal{S}_0} \left[\dot{\vec{a}} + \vec{a} \vec{\nabla}_{\vec{x}} \cdot \vec{v} - \hat{L} \vec{a} \right] \cdot J \hat{F}^{-T} \vec{N} dS \end{aligned} \tag{2.38}$$

where the relation $\dot{\hat{F}} = \hat{L} \hat{F}$ has been used. Finally, coming back to the current configuration we obtain

$$\frac{d}{dt} \int_{\mathcal{S}_t} \vec{a} \cdot \vec{n} ds = \int_{\mathcal{S}_t} \left[\dot{\vec{a}} + \vec{a} \vec{\nabla}_{\vec{x}} \cdot \vec{v} - \hat{L} \vec{a} \right] \cdot \vec{n} ds \tag{2.39}$$

Since the material derivative is given by $\dot{\vec{a}} = \frac{\partial \vec{a}}{\partial t} + \frac{\partial \vec{a}}{\partial \vec{x}} \cdot \vec{v}$, we obtain

$$\frac{d}{dt} \int_{\mathcal{S}_t} \vec{a} \cdot \vec{n} ds = \int_{\mathcal{S}_t} \left[\frac{\partial \vec{a}}{\partial t} + \frac{\partial \vec{a}}{\partial \vec{x}} \cdot \vec{v} + \vec{a} \vec{\nabla}_{\vec{x}} \cdot \vec{v} - \hat{L} \vec{a} \right] \cdot \vec{n} ds \tag{2.40}$$

It is simple to verify that $\vec{\nabla}_{\vec{x}} \wedge (\vec{a} \wedge \vec{v}) + \vec{v} \vec{\nabla}_{\vec{x}} \cdot \vec{a} = \frac{\partial \vec{a}}{\partial \vec{x}} \cdot \vec{v} + \vec{a} \vec{\nabla}_{\vec{x}} \cdot \vec{v} - \hat{L} \vec{a}$ and therefore we can write

$$\frac{d}{dt} \int_{\mathcal{S}_t} \vec{a} \cdot \vec{n} ds = \int_{\mathcal{S}_t} \left[\frac{\partial \vec{a}}{\partial t} + \vec{\nabla}_{\vec{x}} \wedge (\vec{a} \wedge \vec{v}) + \vec{v} \vec{\nabla}_{\vec{x}} \cdot \vec{a} \right] \cdot \vec{n} ds \tag{2.41}$$

The Nanson relation $\vec{n} ds = J \hat{F}^{-T} \vec{N} dS$ can be also applied in order to obtain the so-called Piola identity. To this aim we use the standard divergence theorem

$$\int_{\partial \mathcal{P}_t} \Psi n_i ds = \int_{\mathcal{P}_t} \frac{\partial \Psi}{\partial x_i} d\vec{x} \tag{2.42}$$

if $\Psi = 1$ identically, we obtain $\int_{\partial \mathcal{P}_t} n_i ds = 0$ and, therefore

$$\int_{\partial \mathcal{P}_t} \vec{n} ds = \int_{\partial \mathcal{P}_0} J \hat{F}^{-T} \vec{N} dS = \int_{\mathcal{P}_0} \vec{\nabla}_{\vec{X}} \cdot \left(J \hat{F}^{-1} \right) d\vec{X} = 0 \tag{2.43}$$

which means

$$\vec{\nabla}_{\vec{X}} \cdot \left(J \hat{F}^{-1} \right) = 0 \ \Rightarrow \frac{\partial}{\partial X_j} \left(J \frac{\partial X_j}{\partial x_i} \right) = 0 \tag{2.44}$$

This relation will be useful to obtain the balance equations of the continuum mechanics in the Lagrangian description.

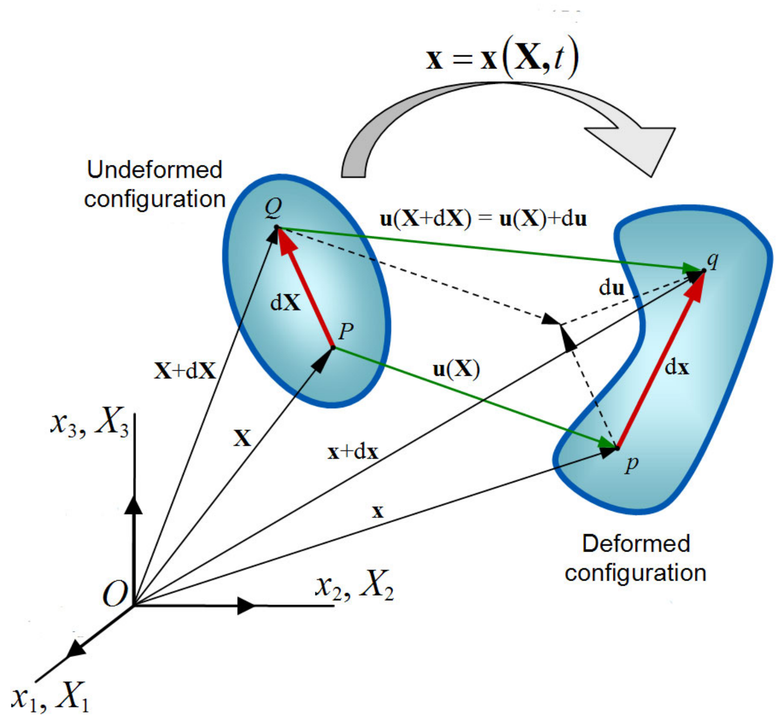

Figure 2. Infinitesimal vector $d\vec{X}$ in Ω_0 and its deformed version $d\vec{x}$ in Ω_t.

3. Strain

The measure of the deformation between the reference and the current configuration is an important topic in continuum mechanics and it can be performed in several ways. The starting quantity is the deformation gradient $\hat{F}(\vec{X})$ (in the Lagrangian formalism) or its inverse $\hat{G}(\vec{x})$ (in the Eulerian formalism). We consider two infinitesimal vectors $d\vec{X}$ and $d\vec{Y}$ in Ω_0 and their deformed versions $d\vec{x}$ and $d\vec{y}$ in Ω_t (see Fig. 2 for the deformation of $d\vec{X}$). The changes of lengths and angles are controlled by the scalar product of the vectors and, therefore, we define the right and the left Cauchy tensors $\hat{C}$ and $\hat{B}$ in order to obtain $d\vec{x} \cdot d\vec{y} = d\vec{X} \cdot \hat{C} d\vec{Y}$ or $d\vec{X} \cdot d\vec{Y} = d\vec{x} \cdot \hat{B}^{-1} d\vec{y}$ (see Table 1). The variations of the scalar product (moving from the reference to the current configuration) are described by the Green-Lagrange tensor $\hat{\eta}$ and by the Almansi-Eulero tensor $\hat{e}$ as summarized in Table 1.

Moreover, the gradients of the displacements field $\vec{u}(\vec{X})$ and $\vec{u}(\vec{x})$ are defined by $\hat{J}_L = \frac{\partial \vec{u}}{\partial \vec{X}}$ and $\hat{J}_E = \frac{\partial \vec{u}}{\partial \vec{x}}$ in the Lagrangian and Eulerian vision, respectively. In Table 2 we can find: (i) the effective variation of length for the vector $d\vec{X} = \vec{N}\|d\vec{X}\|$ deformed into $d\vec{x} = \vec{n}\|d\vec{x}\|$; (ii) the variation of the right angle between the unit vectors $\vec{N}$ and $\vec{T}$ ($\vec{N} \cdot \vec{T} = 0$ in Ω_0) deformed into $\vec{n}$ and $\vec{t}$ (in Ω_t): θ_{nt} is the angle in Ω_t and, therefore, $\gamma_{NT} = \frac{\pi}{2} - \theta_{nt}$ is the angle variation (with opposite sign); (iii) the variation of the right angle between the unit vectors $\vec{n}$ and $\vec{t}$ ($\vec{n} \cdot \vec{t} = 0$ in Ω_t) originally placed at $\vec{N}$ and $\vec{T}$ (in Ω_0): θ_{NT} is the angle in Ω_0 and, therefore, $\gamma_{nt} = \theta_{NT} - \frac{\pi}{2}$ is the angle variation (with

Table 1. Strains definitions and properties in Lagrangian and Eulerian formalisms.

Lagrangian vision	Eulerian vision
Right Cauchy tensor $\hat{C}=\hat{F}^T\hat{F}$ $\hat{C}^{-1}=\hat{G}\hat{G}^T$ $d\vec{x}\cdot d\vec{y}=d\vec{X}\cdot\hat{C}d\vec{Y}$	Left Cauchy tensor $\hat{B}=\hat{F}\hat{F}^T$ $\hat{B}^{-1}=\hat{G}^T\hat{G}$ $d\vec{X}\cdot d\vec{Y}=d\vec{x}\cdot\hat{B}^{-1}d\vec{y}$
Green-Lagrange tensor $\hat{\eta}=\frac{1}{2}\left(\hat{C}-\hat{I}\right)$ $d\vec{x}\cdot d\vec{y}-d\vec{X}\cdot d\vec{Y}=2d\vec{X}\cdot\hat{\eta}d\vec{Y}$	Almansi-Eulero tensor $\hat{e}=\frac{1}{2}\left(\hat{I}-\hat{B}^{-1}\right)$ $d\vec{x}\cdot d\vec{y}-d\vec{X}\cdot d\vec{Y}=2d\vec{x}\cdot\hat{e}d\vec{y}$
Lagrange displacement gradient $\hat{J}_L=\frac{\partial\vec{u}}{\partial\vec{X}}$ $\hat{F}=\hat{I}+\hat{J}_L$ $\hat{C}=\hat{I}+\hat{J}_L+\hat{J}_L^T+\hat{J}_L^T\hat{J}_L$ $\hat{\eta}=\frac{1}{2}\left(\hat{J}_L+\hat{J}_L^T+\hat{J}_L^T\hat{J}_L\right)$	Eulero displacement gradient $\hat{J}_E=\frac{\partial\vec{u}}{\partial\vec{x}}$ $\hat{F}^{-1}=\hat{I}-\hat{J}_E$ $\hat{B}^{-1}=\hat{I}-\hat{J}_E-\hat{J}_E^T+\hat{J}_E^T\hat{J}_E$ $\hat{e}=\frac{1}{2}\left(\hat{J}_E+\hat{J}_E^T-\hat{J}_E^T\hat{J}_E\right)$

opposite sign); (iv) the variations of volume and surface measures.

Any non singular tensor (describing a deformation) can be decomposed in two different ways

$$\hat{F}=\hat{R}\,\hat{U}=\hat{V}\hat{R} \tag{3.1}$$

where $\hat{R}$ is a rotation matrix ($\hat{R}\hat{R}^T=\hat{R}^T\hat{R}=\hat{I}$) while $\hat{U}$ and $\hat{V}$ are symmetric and positive definite tensors. In order to prove this polar decomposition theorem due to Cauchy, we use the right Cauchy tensor $\hat{C}=\hat{F}^T\hat{F}$: it is symmetric since $\left(\hat{F}^T\hat{F}\right)^T=\hat{F}^T\hat{F}^{TT}=\hat{F}^T\hat{F}$ and it is positive definite as proved by the following relation

$$\vec{w}^T\hat{F}^T\hat{F}\vec{w}=\left(\hat{F}\vec{w}\right)^T\left(\hat{F}\vec{w}\right)=\|\hat{F}\vec{w}\|\geq 0\ \forall\ \vec{w} \tag{3.2}$$

If $\hat{F}^T\hat{F}$ is symmetric and positive definite then it can be diagonalized in the field of real numbers. Therefore, we can write $\hat{F}^T\hat{F}=\hat{Q}^{-1}\hat{\triangle}\hat{Q}$ where $\hat{Q}$ is non singular and $\hat{\triangle}$ is diagonal. We define

$$\hat{U}=\sqrt{\hat{F}^T\hat{F}}=\sqrt{\hat{C}} \tag{3.3}$$

The square root of the tensor can be defined (and calculated) as follows

$$\hat{U}=\sqrt{\hat{F}^T\hat{F}}=\sqrt{\hat{Q}^{-1}\hat{\triangle}\hat{Q}}=\hat{Q}^{-1}\sqrt{\hat{\triangle}}\hat{Q} \tag{3.4}$$

in fact

$$\left(\hat{Q}^{-1}\sqrt{\hat{\triangle}}\hat{Q}\right)^2=\hat{Q}^{-1}\sqrt{\hat{\triangle}}\hat{Q}\hat{Q}^{-1}\sqrt{\hat{\triangle}}\hat{Q}=\hat{Q}^{-1}\sqrt{\hat{\triangle}}\sqrt{\hat{\triangle}}\hat{Q}=\hat{Q}^{-1}\hat{\triangle}\hat{Q} \tag{3.5}$$

Table 2. Variations measure in Lagrangian and Eulerian formalisms.

Lagrangian vision	Eulerian vision
Lagrangian length variation $\vec{N}=\frac{d\vec{X}}{\|d\vec{X}\|}$ $\epsilon_{NN}=\frac{\|d\vec{x}\|-\|d\vec{X}\|}{\|d\vec{X}\|}=\sqrt{\vec{N}\cdot\hat{C}\vec{N}}-1$ $\epsilon_{NN}+\frac{1}{2}\epsilon_{NN}^2=\vec{N}\cdot\hat{\eta}\vec{N}$	Eulerian length variation $\vec{n}=\frac{d\vec{x}}{\|d\vec{x}\|}$ $\epsilon_{nn}=\frac{\|d\vec{x}\|-\|d\vec{X}\|}{\|d\vec{x}\|}=1-\sqrt{\vec{n}\cdot\hat{B}^{-1}\vec{n}}$ $\epsilon_{nn}-\frac{1}{2}\epsilon_{nn}^2=\vec{n}\cdot\hat{e}\vec{n}$
Lagrangian angle variation $\vec{N}\cdot\vec{T}=0$ $\gamma_{NT}=\frac{\pi}{2}-\theta_{nt}$ $\sin(\gamma_{NT})=\frac{2\vec{N}\cdot\hat{\eta}\vec{T}}{\sqrt{\vec{N}\cdot\hat{C}\vec{N}}\sqrt{\vec{T}\cdot\hat{C}\vec{T}}}$	Eulerian angle variation $\vec{n}\cdot\vec{t}=0$ $\gamma_{nt}=\theta_{NT}-\frac{\pi}{2}$ $\sin(\gamma_{nt})=\frac{2\vec{n}\cdot\hat{e}\vec{t}}{\sqrt{\vec{n}\cdot\hat{B}^{-1}\vec{n}}\sqrt{\vec{t}\cdot\hat{B}^{-1}\vec{t}}}$
Lagrangian volume variation $J=\det(\hat{F})$ $\Theta_V=\frac{dv-dV}{dV}=J-1$	Eulerian volume variation $J^{-1}=\det(\hat{G})$ $\Theta_v=\frac{dv-dV}{dv}=1-\frac{1}{J}$
Lagrangian surface variation $\vec{N}dS=J^{-1}\hat{F}^T\vec{n}ds$ $\Theta_N=\frac{\|\vec{n}ds\|-\|\vec{N}dS\|}{\|\vec{N}dS\|}$ $\Theta_N=J\sqrt{\vec{N}\cdot\hat{C}^{-1}\vec{N}}-1$	Eulerian surface variation $\vec{n}ds=J\hat{F}^{-T}\vec{N}dS$ $\Theta_n=\frac{\|\vec{n}ds\|-\|\vec{N}dS\|}{\|\vec{n}ds\|}$ $\Theta_n=1-J^{-1}\sqrt{\vec{n}\cdot\hat{B}\vec{n}}$

where $\sqrt{\hat{\triangle}}=\text{diag}(\sqrt{\lambda_i})$ if $\hat{\triangle}=\text{diag}(\lambda_i)$ (the symbol diag explicitely indicates the entries of a diagonal matrix). Finally, we define $\hat{R}=\hat{F}\hat{U}^{-1}$ and we verify its ortogonality

$$\hat{R}^T\hat{R}=\left(\hat{U}^{-1}\right)^T\hat{F}^T\hat{F}\hat{U}^{-1}=\left(\hat{U}^{-1}\right)^T\hat{U}^2\hat{U}^{-1}=\hat{U}^{-1}\hat{U}\hat{U}\hat{U}^{-1}=\hat{I} \tag{3.6}$$

This concludes the proof of the first polar decomposition. We have to prove the unicity of the right decomposition $\hat{F}=\hat{R}\,\hat{U}$. We can suppose the two different decompositions $\hat{F}=\hat{R}\,\hat{U}=\hat{R}^*\hat{U}^*$ exist. It follows that $\hat{F}^T\hat{F}=\hat{U}^2=\hat{U}^{*2}$ from which $\hat{U}=\hat{U}^*$ and, therefore, $\hat{R}=\hat{R}^*$. It proves the unicity of the right decomposition. Similarly we can obtain the left decomposition by defining $\hat{V}=\sqrt{\hat{F}\hat{F}^T}=\sqrt{\hat{B}}$: it is possible to prove that it is symmetric and positive definite and we define $\hat{R}'=\hat{V}^{-1}\hat{F}$, which is orthogonal.

To conclude we must verify that $\hat{R}'=\hat{R}$. Since $\hat{R}'\left(\hat{R}'\right)^T=\hat{I}$ we have $\hat{F}=\hat{V}\hat{R}'=\hat{R}'\left(\hat{R}'\right)^T\hat{V}\hat{R}'$. The unicity of the right decomposition ($\hat{F}=\hat{R}\,\hat{U}$) allows us to affirm that $\hat{R}'=\hat{R}$ and that $\hat{U}=\hat{R}^T\hat{V}\hat{R}$. This completes the proof of the polar decomposition Cauchy theorem.

This decomposition implies that the deformation of a line element $d\vec{X}$ in the undeformed configuration onto $d\vec{x}$ in the deformed configuration, i.e. $d\vec{x}=\hat{F}d\vec{X}$ may be obtained either by first stretching the element by $\hat{U}$ i.e. $d\vec{x}'=\hat{U}d\vec{X}$, followed by

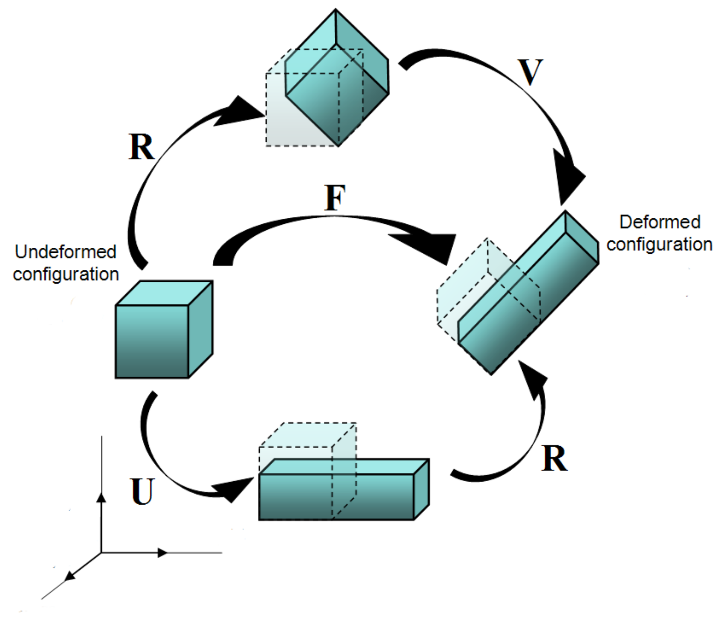

Figure 3. Polar decomposition applied to a given deformation.

a rotation $\hat{R}$, i.e. $d\vec{x} = \hat{R}d\vec{x}'$ or, equivalently, by applying a rigid rotation $\hat{R}$ first, i.e. $d\vec{x}'' = \hat{R}d\vec{X}$ followed later by a stretching $\hat{V}$, i.e. $d\vec{x} = \hat{V}d\vec{x}''$ (seeFig. 3).

4. Stress

In continuum mechanics we must consider two systems of forces acting on a given region of a material body:

- the *body forces*. They are dependent on the external fields acting on the elastic body and they are described by the vector field $\vec{b}(\vec{x})$ representing their density on the volume in the current configuration. The physical meaning of such a density of forces can be summed up stating that the total force $d\vec{F}_v$ applied to a small volume $d\vec{x}$ centered on the point $\vec{x}$ is given by $d\vec{F}_v = \vec{b}(\vec{x})d\vec{x}$. A typical example is given by the gravitational forces proportional to the mass of the region under consideration. In this case we can write $d\vec{F}_v = \vec{g}dm$ where $\vec{g}$ is the gravitational acceleration and dm is the mass of the volume $d\vec{x}$. If we define $\rho = \frac{dm}{d\vec{x}}$ as the density of the body, we simply obtain $\vec{b}(\vec{x}) = \rho\vec{g}$.
- the *surface forces*. In continuum mechanics we are additionally concerned with the interaction between neighbouring portions of the interiors of deformable bodies. In reality such an interaction consists of complex interatomic forces, but we make the simplifying assumption that the effect of all such forces across any given surface may be adequately represented by a single vector field defined over the surface. It is important to observe that the nature of the forces exerted between two bodies in contact is identical to the nature of the actions applied between two portions of

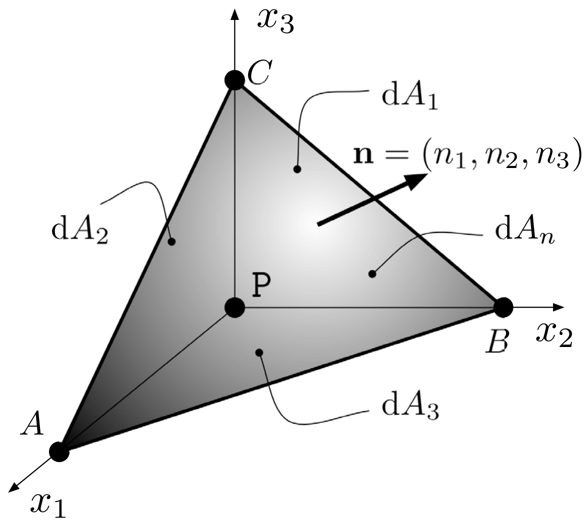

Figure 4. Cauchy tetrahedron on a generic point P.

the same body, separated by an ideal surface.

In order to begin the mathematical descriptions of the forces, it is useful to introduce the following notation for the surface force $d\vec{F}_s$ applied to the area element ds (with unit normal vector $\vec{n}$) of the deformed configuration

$$d\vec{F}_s = \vec{f}\,(\vec{x}, \vec{n}, t)\, ds \tag{4.1}$$

where $\vec{f}$ assumes the meaning of a density of forces distributed over the surface. By definition, the force $d\vec{F}_s$ is applied by the region where the unit vector $\vec{n}$ is directed to the other region beyond the ideal surface (or interface). We can now recall the Cauchy theorem on the existence of the stress tensor describing the distribution of the surface forces in a given elastic body. More precisely, we can say that a tensor $\hat{T}$ exists such that

$$\vec{f}\,(\vec{x}, \vec{n}, t) = \hat{T}(\vec{x}, t)\vec{n} \tag{4.2}$$

where $\vec{n}$ is the external normal unit vector to the surface delimiting the portion of body subjected to the force field $\vec{f}$. The quantity $\hat{T}$ has been called Cauchy stress tensor or simply stress tensor. This very important result has been firstly published by Cauchy in 1827 in the text "Exercices de mathématique". The forces applied to the area element can be therefore written in the following form

$$d\vec{F}_s = \hat{T}(\vec{x})\vec{n} ds \tag{4.3}$$

or, considering the different components $\frac{dF_{s,i}}{ds} = T_{ij}n_j$. So, we may identify the stress tensor $\hat{T}$ with a sort of vector pressure. Its physical unit is therefore the Pa (typical values in solid mechanics range from MPa to GPa). The proof of the Cauchy theorem can be performed as follows.

We consider a generic point P in the deformed configuration and a small tetrahedron as described in Fig. 4. The oblique plane is defined by a unit vector $\vec{n}$ and by the distance dh from P. The faces of the tetrahedron have areas dA_1, dA_2, dA_3 and dA_n and the outgoing normal unit vectors are $-\vec{E}_1$, $-\vec{E}_2$, $-\vec{E}_3$ and $\vec{n}$ (where the vectors $\vec{E}_i$ belong to the reference base). We define $\vec{f}_1$, $\vec{f}_2$, $\vec{f}_3$ and $\vec{f}_n$ as the surface forces acting on each face and $\vec{b}$ as the body force distributed over the volume. The motion equation is

$$\vec{f}_n \mathrm{d}A_n + \vec{f}_1 \mathrm{d}A_1 + \vec{f}_2 \mathrm{d}A_2 + \vec{f}_3 \mathrm{d}A_3 + \vec{b}\, \mathrm{d}v = \rho \vec{a} \mathrm{d}v \tag{4.4}$$

where $\vec{a}$ is the acceleration of the tetrahedron with mass $\rho \mathrm{d}v$. From Eq.(4.1) we can identify $\vec{f}_n = \vec{f}(\vec{n}, \vec{x}, t)$ and $\vec{f}_k = \vec{f}\left(-\vec{E}_k, \vec{x}, t\right)$ $, \forall\ k = 1, 2, 3$. Moreover, $\mathrm{d}A_i = n_i \mathrm{d}A_n$ $, \forall\ i = 1, 2, 3$ and $\mathrm{d}v = \frac{1}{3}\mathrm{d}A_n \mathrm{d}h$, so we can write Eq. (4.4) as follows (sum over j)

$$\vec{f}(\vec{n}, \vec{x}, t) + \vec{f}\left(-\vec{E}_j, \vec{x}, t\right) n_j + \frac{1}{3}\, \vec{b}\, \mathrm{d}h = \frac{1}{3}\, \rho\, \vec{a}\, \mathrm{d}h \tag{4.5}$$

In the limit of d$h \to 0$ we obtain (sum over j)

$$\vec{f}(\vec{n}, \vec{x}, t) = -\vec{f}\left(-\vec{E}_j, \vec{x}, t\right) n_j \tag{4.6}$$

Now we can use the previous result with $\vec{n} = \vec{E}_i$ (for any $i = 1, 2, 3$), by obtaining

$$\vec{f}\left(\vec{E}_i, \vec{x}, t\right) = -\vec{f}\left(-\vec{E}_i, \vec{x}, t\right) \tag{4.7}$$

This is a sort of third law of the dynamics written in term of surface forces. Now, Eq.(4.6) can be simply rewritten as (sum over j)

$$\vec{f}(\vec{n}, \vec{x}, t) = \vec{f}\left(\vec{E}_j, \vec{x}, t\right) n_j \tag{4.8}$$

This result shows that the surface force $\vec{f}$ on a given plane is determined by the three surface forces on the three coordinate planes; in components

$$f_i(\vec{n}, \vec{x}, t) = \vec{f}(\vec{n}, \vec{x}, t) \cdot \vec{E}_i = \vec{f}\left(\vec{E}_j, \vec{x}, t\right) \cdot \vec{E}_i n_j = T_{ij} n_j \tag{4.9}$$

where the Cauchy stress $\hat{T}$ is represented by $T_{ij} = \vec{f}\left(\vec{E}_j, \vec{x}, t\right) \cdot \vec{E}_i$. To better understand the physical meaning of the stress tensor we consider the cubic element of volume shown in Fig.5, corresponding to an infinitesimal portion $\mathrm{d}V = (\mathrm{d}l)^3$ taken in an arbitrary solid body. The six faces of the cube have been numbered as shown in Fig.5. We suppose that a stress $\hat{T}$ is applied to that region: the T_{ij} component represents the pressure applied on the j-th face along the i-th direction.

The Cauchy stress tensor is the most natural and physical measure of the state of stress at a point in the deformed configuration and measured per unit area of the deformed configuration. It is the quantity most commonly used in spatial or Eulerian description of problems in continuum mechanics. Some other stress measures must be introduced in order to describe continuum mechanics in the Lagrangian formalism. From Cauchy formula, we have $d\vec{F}_s = \hat{T}\vec{n}ds$, where $\hat{T}$ is the Cauchy stress tensor. In a similar fashion, we introduce a stress tensor $\hat{T}^{1\mathcal{PK}}$, called the first Piola-Kirchhoff stress tensor, such that $d\vec{F}_s = \hat{T}^{1\mathcal{PK}}\vec{N}dS$. By using the Nanson formula $\vec{n}ds = J\hat{F}^{-T}\vec{N}dS$ we obtain

$$d\vec{F}_s = \hat{T}J\hat{F}^{-T}\vec{N}dS = \hat{T}^{1\mathcal{PK}}\vec{N}dS \tag{4.10}$$

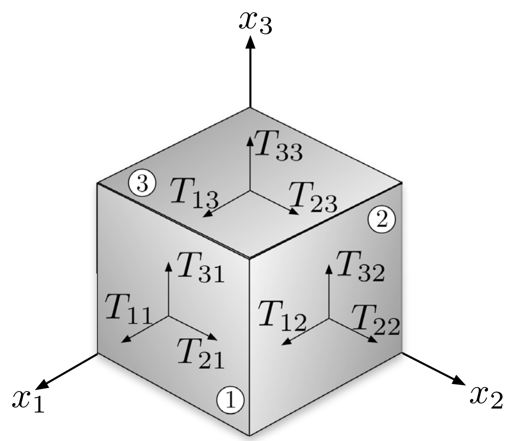

Figure 5. Geometrical representation of the stress tensor $\hat{T}$: the T_{ij} component represents the pressure applied on the j-th face of the cubic volume along the i-th direction.

and therefore

$$\hat{T}^{1\mathcal{PK}} = J\hat{T}\hat{F}^{-T} \tag{4.11}$$

Sometimes it is useful to introduce another state of stress $\hat{T}^{2\mathcal{PK}}$, called the second Piola-Kirchhoff stress tensor, defined as $\hat{F}^{-1}d\vec{F}_s = \hat{T}^{2\mathcal{PK}}\vec{N}dS$. We simply obtain

$$\hat{F}^{-1}d\vec{F}_s = \hat{F}^{-1}\hat{T}J\hat{F}^{-T}\vec{N}dS = \hat{T}^{2\mathcal{PK}}\vec{N}dS \tag{4.12}$$

and therefore

$$\hat{T}^{2\mathcal{PK}} = J\hat{F}^{-1}\hat{T}\hat{F}^{-T} = \hat{F}^{-1}\hat{T}^{1\mathcal{PK}} \tag{4.13}$$

The stress tensors $\hat{T}^{1\mathcal{PK}}$ and $\hat{T}^{2\mathcal{PK}}$ will be very useful for the finite elasticity theory described within the Lagrangian formalism.

5. Continuity equation

The first balance equation of the continuum mechanics concerns the mass distribution. We define the mass density: we will use $\rho_0(\vec{X})$ in the Lagrangian formalism and $\rho\,(\vec{x}, t)$ in the Eulerian description. The total mass of the region $\mathcal{P}_t$ is given by

$$m\,(\mathcal{P}_t) = \int_{\mathcal{P}_t} \rho(\vec{x}, t)d\vec{x} \tag{5.1}$$

The consevation of the mass gives

$$\int_{\mathcal{P}_t} \rho(\vec{x}, t)d\vec{x} = \int_{\mathcal{P}_0} \rho_0(\vec{X})d\vec{X} \text{ or } \frac{d}{dt}\int_{\mathcal{P}_t} \rho(\vec{x}, t)d\vec{x} = 0 \tag{5.2}$$

The first equality in Eq.(5.2) can be also written

$$\int_{\mathcal{P}_0} \rho J d\vec{X} = \int_{\mathcal{P}_0} \rho_0 d\vec{X} \tag{5.3}$$

and we simply obtain

$$\rho J = \rho_0 \tag{5.4}$$

On the other hand, from the second equality in Eq.(5.2) we have

$$\int_{\mathcal{P}_t} \left(\dot{\rho} + \rho \vec{\nabla}_{\vec{x}} \cdot \vec{v}\right) d\vec{x} = \int_{\mathcal{P}_t} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}_{\vec{x}} \cdot (\rho \vec{v})\right] d\vec{x} = 0 \tag{5.5}$$

and therefore we obtain two forms of the continuity equation

$$\dot{\rho} + \rho \vec{\nabla}_{\vec{x}} \cdot \vec{v} = 0 \tag{5.6}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_{\vec{x}} \cdot (\rho \vec{v}) = 0 \tag{5.7}$$

It is important for the following applications to evaluate expressions of this kind: $\frac{d}{dt}\int_{\mathcal{P}_t} \rho(\vec{x},t)\Psi(\vec{x},t)d\vec{x}$; to this aim we use the Reynolds theorem with $\phi = \rho\Psi$

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\Psi d\vec{x} = \int_{\mathcal{P}_t} \left(\dot{\rho}\Psi + \rho\dot{\Psi} + \rho\Psi\vec{\nabla}_{\vec{x}} \cdot \vec{v}\right) d\vec{x} = \int_{\mathcal{P}_t} \rho\dot{\Psi} d\vec{x} \tag{5.8}$$

It means that, when there is the density in the integrand, the time derivative must be applied directly to the function Ψ.

6. Balance equations: Euler description

The other two important balance equations can be derived by the principles of linear and angular momentum. When dealing with a system of particles, we can deduce from Newton's laws of motion that the resultant of the external forces is equal to the rate of change of the total linear momentum of the system. By taking moments about a fixed point, we can also show that the resultant moment of the external forces is equal to the rate of change of the total moment of momentum. Here we define the linear and angular momentum density for a continuum and we introduce balance laws for these quantities. We consider a portion $\mathcal{P}_t$ in a material body and we define $\vec{P}$ as its linear momentum, $\vec{F}$ as the resultant of the applied forces, $\vec{L}$ as the total angular momentum and, finally, $\vec{M}$ as the resultant moment of the applied forces. The standard principles for a system of particles can be written as follows

$$\frac{d\vec{P}}{dt} = \vec{F} \qquad \frac{d\vec{L}}{dt} = \vec{M} \tag{6.1}$$

We start with the first principle, applied to the portion of body contained to the region $\mathcal{P}_t$, limited by the closed surface $\partial\mathcal{P}_t$

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\vec{v} d\vec{x} = \int_{\partial\mathcal{P}_t} \hat{T}\vec{n} ds + \int_{\mathcal{P}_t} \vec{b} d\vec{x} \tag{6.2}$$

where we have utilized the decomposition of the forces (body forces and surface forces) as described in the previous section. The previous equation can be simplified by means of Eq.(5.8) and the divergence theorem, by obtaining

$$\int_{\mathcal{P}_t} \rho\dot{\vec{v}} d\vec{x} = \int_{\mathcal{P}_t} \vec{\nabla}_{\vec{x}} \cdot \hat{T} d\vec{x} + \int_{\mathcal{P}_t} \vec{b} d\vec{x} \tag{6.3}$$

Since the volume $\mathcal{P}_t$ is arbitrary, we easily obtain the first balance equation for the elasticty theory (Eulerian description)

$$\vec{\nabla}_{\vec{x}} \cdot \hat{T} + \vec{b} = \rho\dot{\vec{v}} \tag{6.4}$$

This is the basic linear momentum equation of continuum mechanics. We remark that the divergence of a tensor is applied on the second index; in fact, in components, we simply obtain

$$\frac{\partial T_{ji}}{\partial x_i} + b_j = \rho\dot{v}_j \tag{6.5}$$

Further, we observe that

$$\dot{\vec{v}} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial \vec{x}} \cdot \vec{v} = \frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla}_{\vec{x}}\left(\vec{v}\cdot\vec{v}\right) + \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v} \tag{6.6}$$

and, therefore Eq.(6.4) is equivalent to

$$\vec{\nabla}_{\vec{x}} \cdot \hat{T} + \vec{b} = \rho\left[\frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial \vec{x}} \cdot \vec{v}\right] \tag{6.7}$$

or

$$\vec{\nabla}_{\vec{x}} \cdot \hat{T} + \vec{b} = \rho\left[\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla}_{\vec{x}}\left(\vec{v}\cdot\vec{v}\right) + \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}\right] \tag{6.8}$$

Now, we consider the principle of the angular momentum. For the region $\mathcal{P}_t$ such a balance equation can be written in the following form

$$\frac{d}{dt}\int_{\mathcal{P}_t} \vec{x} \wedge \rho\vec{v} d\vec{x} = \int_{\partial\mathcal{P}_t} \vec{x} \wedge \left(\hat{T}\vec{n}\right) ds + \int_{\mathcal{P}_t} \vec{x} \wedge \vec{b}\, d\vec{x} \tag{6.9}$$

As before, the surface integral can be simplified with the application of the divergence theorem, by obtaining, after some straightforward calculations

$$\int_{\partial\mathcal{P}_t} \vec{x} \times \left(\hat{T}\vec{n}\right) ds = \int_{\mathcal{P}_t} \left[T_{kh} + x_h \frac{\partial T_{kp}}{\partial x_p}\right] \eta_{hkj}\vec{e}_j d\vec{x} \tag{6.10}$$

So, the second balance equation assumes the form

$$\int_{\mathcal{P}_t} \left\{x_h\left[\rho\dot{v}_k - \frac{\partial T_{kp}}{\partial x_p} - b_k\right] - T_{kh}\right\} \eta_{hkj}\vec{e}_j d\vec{x} = 0 \tag{6.11}$$

The term in bracket is zero because of the first balance equation. Therefore, we obtain $\int_{\mathcal{P}_t} T_{kh}\eta_{hkj}\vec{e}_j d\vec{x} = 0$ or, equivalently, $T_{kh}\eta_{hkj} = 0$. Finally, the second principle leads to

$$T_{ij} = T_{ji} \tag{6.12}$$

In other words, we may state that the principle of the angular momentum assures the symmetry of the Cauchy stress tensor.

7. Balance equations: Lagrange description

In finite elasticity theory the Lagrangian description is the most important point of view since it allows to determine the exact transformation $\vec{x} = \mathcal{F}_t\left(\vec{X}\right)$ between the reference and the actual configurations. In the case of finite deformations (arbitrarily large), the Piola-Kirchhoff stress tensors above defined are used to express the stress relative to the reference configuration. This is in contrast to the Cauchy stress tensor which expresses the stress relative to the current configuration. In order to obtain the Lagrangian equations of motion it is useful to introduce the so-called Piola transform $\vec{W}(\vec{X},t)$ (which is a Lagrangian vector field) of a given Eulerian vector field $\vec{w}(\vec{x},t)$

$$\vec{w}(\vec{x},t) \;\Rightarrow\; \vec{W}(\vec{X},t) = J\hat{F}^{-1}\vec{w}(\mathcal{F}_t\left(\vec{X}\right),t) \tag{7.1}$$

An important relation gives the relationship between the divergence of the two fields: of course, the divergence of $\vec{W}(\vec{X},t)$ is calculated with respect to the Lagrangian variables $\vec{X}$ while that of $\vec{w}(\vec{x},t)$ is calculated with respect to the Eulerian variables $\vec{x}$

$$\begin{aligned}\vec{\nabla}_{\vec{X}}\cdot\vec{W}(\vec{X},t) &= \frac{\partial W_i}{\partial X_i} = \frac{\partial}{\partial X_i}\left(J\frac{\partial X_i}{\partial x_s}w_s\right)\\ &= \frac{\partial}{\partial X_i}\left(J\frac{\partial X_i}{\partial x_s}\right)w_s + J\frac{\partial X_i}{\partial x_s}\frac{\partial w_s}{\partial X_i}\end{aligned} \tag{7.2}$$

The first term is zero for the Piola identity given in Eq.(2.44), and therefore

$$\vec{\nabla}_{\vec{X}}\cdot\vec{W}(\vec{X},t) = J\frac{\partial X_i}{\partial x_s}\frac{\partial w_s}{\partial X_i} = J\frac{\partial w_s}{\partial x_s} \tag{7.3}$$

It means that we have obtained the important relation

$$\vec{\nabla}_{\vec{X}}\cdot\vec{W}(\vec{X},t) = J\vec{\nabla}_{\vec{x}}\cdot\vec{w}(\vec{x},t) \tag{7.4}$$

We can also make a Piola transformation on a given index of a tensor. For example, if T_{ji} the Cauchy stress tensor, we may use the above tranformation on the last index. We apply this procedure to transform the motion equation from the Eulerian to the Lagrangian coordinates

$$\frac{\partial T_{ji}}{\partial x_i} + b_j = \rho\dot{v}_j \;\Rightarrow\; \frac{1}{J}\frac{\partial}{\partial X_i}\left(J\frac{\partial X_i}{\partial x_s}T_{js}\right) + b_j = \rho\dot{v}_j \tag{7.5}$$

or, identifying the deformation gradient

$$\frac{\partial}{\partial X_i}\left[J(\hat{F}^{-1})_{is}T_{js}\right] + Jb_j = \rho J\dot{v}_j \tag{7.6}$$

By using the relation $\rho_0 = J\rho$ we obtain

$$\frac{\partial}{\partial X_i}\left[JT_{js}(\hat{F}^{-T})_{si}\right] + \frac{\rho_0}{\rho}b_j = \rho_0\dot{v}_j \tag{7.7}$$

Since we have defined the first Piola-Kirchhoff stress tensor as $\hat{T}^{1\mathcal{PK}} = J\hat{T}\hat{F}^{-T}$ we obtain

$$\vec{\nabla}_{\vec{X}}\cdot\hat{T}^{1\mathcal{PK}} + \frac{\rho_0}{\rho}\vec{b} = \rho_0\dot{\vec{v}} \tag{7.8}$$

Now, we consider the principle of the angular momentum: since $\hat{T} = \frac{1}{J}\hat{T}^{1\mathcal{PK}}\hat{F}^T$ and $\hat{T} = \hat{T}^T$ we obtain

$$\hat{T}^{1\mathcal{PK}}\hat{F}^T = \hat{F}\hat{T}^{1\mathcal{PK}T} \tag{7.9}$$

These two important results can be also expressed in terms of the second Piola-Kirchhoff stress tensor $\hat{T}^{2\mathcal{PK}} = \hat{F}^{-1}\hat{T}^{1\mathcal{PK}}$. We simply obtain the linear momentum balance

$$\vec{\nabla}_{\vec{X}} \cdot \left(\hat{F}\hat{T}^{2\mathcal{PK}}\right) + \frac{\rho_0}{\rho}\vec{b} = \rho_0\dot{\vec{v}} \tag{7.10}$$

and the angular momentum balance

$$\hat{T}^{2\mathcal{PK}} = \hat{T}^{2\mathcal{PK}T} \tag{7.11}$$

Of course, Eqs.(7.10) and (7.11) must be completed by the constitutive equations and by the boundary conditions.

7.1. Novozhilov formulation.

We consider the standard base of unit vectors $\vec{E}_1$, $\vec{E}_2$ and $\vec{E}_3$ in the point $\vec{X}$ of the reference configuration. Since the motion is controlled by the tranformation $\vec{x} = \mathcal{F}_t\left(\vec{X}\right)$, the unit vectors $\vec{e}_i$ in the deformed configuration are given by the direction of the deformed coordinate lines

$$\vec{e}_i = \frac{\frac{\partial \mathcal{F}_t(\vec{X})}{\partial X_i}}{\|\frac{\partial \mathcal{F}_t(\vec{X})}{\partial X_i}\|} = \frac{\hat{F}\vec{E}_i}{\|\hat{F}\vec{E}_i\|} \tag{7.12}$$

We remark that they do not form an orthogonal base. First of all, we simply obtain the norm of $\hat{F}\vec{E}_i$

$$\|\hat{F}\vec{E}_i\| = \sqrt{\left(\hat{F}\vec{E}_i\right)\cdot\left(\hat{F}\vec{E}_i\right)} = \sqrt{F_{ki}F_{ki}} = \sqrt{\left(\hat{F}^T\hat{F}\right)_{ii}} = \sqrt{C_{ii}} \tag{7.13}$$

where $\hat{C}$ is the right Cauchy tensor. We define the unit vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ perpendicular to the planes $(\vec{e}_2, \vec{e}_3)$, $(\vec{e}_1, \vec{e}_3)$ and $(\vec{e}_1, \vec{e}_2)$. It means that we can write

$$\vec{n}_k = \frac{1}{2}\eta_{kij}\frac{\vec{e}_i \wedge \vec{e}_j}{\|\vec{e}_i \wedge \vec{e}_j\|} = \frac{1}{2}\eta_{kij}\frac{\left(\hat{F}\vec{E}_i\right)\wedge\left(\hat{F}\vec{E}_j\right)}{\|\left(\hat{F}\vec{E}_i\right)\wedge\left(\hat{F}\vec{E}_j\right)\|} \tag{7.14}$$

Now, we start with the calculation of $\|\left(\hat{F}\vec{E}_i\right)\wedge\left(\hat{F}\vec{E}_j\right)\|$

$$\begin{aligned}\|\left(\hat{F}\vec{E}_i\right)\wedge\left(\hat{F}\vec{E}_j\right)\| &= \sqrt{\eta_{kst}F_{si}F_{tj}\eta_{kab}F_{ai}F_{bj}} \\ &= \sqrt{\left(\delta_{sa}\delta_{tb} - \delta_{sb}\delta_{ta}\right)F_{si}F_{tj}F_{ai}F_{bj}} \\ &= \sqrt{C_{ii}C_{jj} - C_{ij}^2}\end{aligned} \tag{7.15}$$

We can also write

$$\frac{ds_k}{dS_k} = \sqrt{C_{ii}C_{jj} - C_{ij}^2} \tag{7.16}$$

where the indices i and j are complementary to k and dS_k and ds_k are the surface elements in the reference and current configuration having unit normal vector $\vec{n}_k$. Since $\left(\hat{F}\vec{E}_i\right) \wedge \left(\hat{F}\vec{E}_j\right) = \eta_{qst} F_{si} F_{tj} \vec{E}_q$, we therefore obtain

$$\vec{n}_k = \frac{1}{2}\eta_{kij} \frac{\eta_{qst} F_{si} F_{tj} \vec{E}_q}{\sqrt{C_{ii}C_{jj} - C_{ij}^2}} \tag{7.17}$$

Since $\eta_{qst} F_{si} F_{tj} F_{qa} = J\eta_{aij}$ we can simply write $\eta_{qst} F_{si} F_{tj} = J\eta_{aij}(\hat{F}^{-1})_{aq}$; this result can be used in Eq.(7.17) to yield

$$\vec{n}_k = \frac{1}{2}\eta_{kij} \frac{J\eta_{aij}(\hat{F}^{-1})_{aq}\vec{E}_q}{\sqrt{C_{ii}C_{jj} - C_{ij}^2}} \tag{7.18}$$

When k is fixed the indices i and j can assume two couples of values [if k =1 we have (i,j)=(2,3) or (3,2), if k =2 we have (i,j)=(1,3) or (3,2) and if k =3 we have (i,j)=(2,1) or (1,2)] and the index a must assume the value k. At the end we evetually obtain

$$\vec{n}_k = \frac{J(\hat{F}^{-1})_{kq}\vec{E}_q}{\sqrt{C_{ii}C_{jj} - C_{ij}^2}} = \frac{dS_k}{ds_k} J(\hat{F}^{-1})_{kq}\vec{E}_q \tag{7.19}$$

where the indices i and j are complementary to k (there is not the sum on k). We may consider the forces acting on the three deformed coordinate planes $(\vec{e}_2, \vec{e}_3)$, $(\vec{e}_1, \vec{e}_3)$ and $(\vec{e}_1, \vec{e}_2)$ (having normal unit vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$, respectively) through the expressions

$$\vec{T}\vec{n}_k = \frac{J(\hat{F}^{-1})_{kq}\vec{T}\vec{E}_q}{\sqrt{C_{ii}C_{jj} - C_{ij}^2}} = \frac{dS_k}{ds_k} J(\hat{F}^{-1})_{kq}\vec{T}\vec{E}_q \tag{7.20}$$

These vectors can be represented on both the base $\vec{E}_i$ and $\vec{e}_i$ as follows

$$\vec{T}\vec{n}_k = \sigma_{sk}^E \vec{E}_s \tag{7.21}$$

$$\vec{T}\vec{n}_k = \sigma_{sk}^e \vec{e}_s \tag{7.22}$$

where, since $\vec{E}_1$, $\vec{E}_2$ and $\vec{E}_3$ is an orthonormal base, we have

$$\sigma_{sk}^E = \vec{T}\vec{n}_k \cdot \vec{E}_s = \frac{dS_k}{ds_k} J(\hat{F}^{-1})_{kq}\vec{T}\vec{E}_q \cdot \vec{E}_s = \frac{dS_k}{ds_k} J(\hat{F}^{-1})_{kq} T_{sq} \tag{7.23}$$

Moreover, we have the following relation between σ_{sk}^E and σ_{sk}^e

$$\sigma_{sk}^E = \vec{T}\vec{n}_k \cdot \vec{E}_s = \sigma_{jk}^e \vec{e}_j \cdot \vec{E}_s = \sigma_{jk}^e \frac{\hat{F}\vec{E}_j \cdot \vec{E}_s}{\sqrt{C_{jj}}} = \frac{1}{\sqrt{C_{jj}}} F_{sj}\sigma_{jk}^e \tag{7.24}$$

The representations σ_{sk}^E and σ_{sk}^e have been introduced by Novozhilov in his pioneering book on nonlinear elasticity. The Lagrangian equation of motion can be written as (see Eq.(7.6))

$$\frac{\partial}{\partial X_k}\left[J(\hat{F}^{-1})_{kq} T_{sq}\right] + Jb_s = \rho J \dot{v}_s \tag{7.25}$$

and then it can be expressed in terms of σ^E_{sk}

$$\frac{\partial}{\partial X_k}\left[\frac{ds_k}{dS_k}\sigma^E_{sk}\right]+Jb_s=\rho J\dot{v}_s \tag{7.26}$$

or in terms of σ^e_{sk}

$$\frac{\partial}{\partial X_k}\left[\frac{ds_k}{dS_k}\frac{1}{\sqrt{C_{jj}}}F_{sj}\sigma^e_{jk}\right]+Jb_s=\rho J\dot{v}_s \tag{7.27}$$

Finally, since it is evident that $\sqrt{C_{jj}}=dl_j/dL_j$, we can state the Lagrangian equations of motion in the Novozhilov form

$$\frac{\partial}{\partial X_k}\left[\frac{\frac{ds_k}{dS_k}}{\frac{dl_j}{dL_j}}F_{sj}\sigma^e_{jk}\right]+Jb_s=\rho J\dot{v}_s \tag{7.28}$$

8. Nonlinear constitutive equations

The constitutive equations represent the relation between the stress and the strain and, therefore, they depend on the material under consideration. Here we prove that there is a strong conceptual connection between the constitutive equations and the energy balance for a continuum body. We start from the motion equation in the Eulerian formalism and we multiply both sides to the velocity component v_j

$$v_j\frac{\partial T_{ji}}{\partial x_i}+v_jb_j=\rho v_j\dot{v}_j \tag{8.1}$$

This expression can also be written as

$$\frac{\partial\left(v_jT_{ji}\right)}{\partial x_i}-T_{ji}\frac{\partial v_j}{\partial x_i}+v_jb_j=\rho v_j\dot{v}_j \tag{8.2}$$

The Eulerian velocity gradient $L_{ji}=\frac{\partial v_j}{\partial x_i}$ can be decomposed in the symmetric and skew-symmetric parts

$$L_{ji}=\frac{\partial v_j}{\partial x_i}=\underbrace{\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)}_{symmetric}+\underbrace{\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)}_{skew-symmetric}=D_{ji}+W_{ji} \tag{8.3}$$

where $\hat{D}$ is the rate of deformation tensor and $\hat{W}$ is the spin tensor. Therefore, the energy balance equation assumes the local form form

$$\frac{\partial\left(v_jT_{ji}\right)}{\partial x_i}-T_{ji}D_{ji}+v_jb_j=\rho v_j\dot{v}_j \tag{8.4}$$

By using the property in Eq. (5.8) we also obtain the global version on the region $\mathcal{P}_t$

$$\frac{d}{dt}\int_{\mathcal{P}_t}\frac{1}{2}\rho v_jv_jd\vec{x}+\int_{\mathcal{P}_t}T_{ji}D_{ji}d\vec{x}=\int_{\partial\mathcal{P}_t}T_{ji}n_iv_jd\vec{x}+\int_{\mathcal{P}_t}v_jb_jd\vec{x} \tag{8.5}$$

The second side of this balance represents the power input (product between force and velocity) consisting of the rate of work done by external surface tractions $T_{ji}n_i$ per unit area and body forces b_j per unit volume of the region $\mathcal{P}_t$ bounded by $\partial\mathcal{P}_t$. Since the time-rate of change of the total energy is equal to the the rate of work done by the external forces (first principle of thermodynamics without thermal effects), we identify the first side as $d\mathcal{E}/dt$ where $\mathcal{E}$ is the total energy contained in $\mathcal{P}_t$. Moreover, the total energy can be written as $\mathcal{E} = \mathcal{K}+\mathcal{U}$ where $\mathcal{K}$ is the kinetic energy and $\mathcal{U}$ is the potential energy. Since $\mathcal{K} = \int_{\mathcal{P}_t} \frac{1}{2}\rho v_j v_j d\vec{x}$ is the standard kinetic energy, we identify

$$\frac{d\mathcal{U}}{dt} = \int_{\mathcal{P}_t} T_{ji} D_{ji} d\vec{x} \tag{8.6}$$

We define the energy density U per unit volume in the reference configuration and therefore $\frac{\rho}{\rho_0}U$ is the energy density per unit volume in the current configuration. We obtain

$$\mathcal{U} = \int_{\mathcal{P}_t} \frac{\rho}{\rho_0} U d\vec{x} \tag{8.7}$$

By drawing a comparison between Eqs.(8.6) and (8.7) we obtain

$$\int_{\mathcal{P}_t} T_{ji} D_{ji} d\vec{x} = \frac{d}{dt}\int_{\mathcal{P}_t} \frac{\rho}{\rho_0} U d\vec{x} \tag{8.8}$$

By using the property in Eq. (5.8) we obtain

$$\frac{\rho}{\rho_0}\dot{U} = T_{ji} D_{ji} \tag{8.9}$$

We introduce now a general statement affirming that the strain energy function U depends upon the deformation gradient $\hat{F}$: therefore, we have $U = U(\hat{F})$. This relation can be simplified by means of the principle of material objectivity (or material frame indifference), which says that the energy (and the stress) in the body should be the same regardless of the reference frame from which it is measured. If we consider a motion $\vec{x} = \mathcal{F}_t(\vec{X})$ we obtain a corresponding deformation gradient $\hat{F}$; on the other hand, if we consider a roto-translated motion $\vec{x} = \hat{Q}(t)\mathcal{F}_t(\vec{X}) + \vec{c}(t)$ (where $\hat{Q}(t)$ is an orthogonal matrix and $\vec{c}(t)$ is an arbitrary vector), then the deformation gradient is $\hat{Q}\hat{F}$. In both cases we must have the same energy and therefore

$$U(\hat{F}) = U(\hat{Q}\hat{F}) \ \forall \hat{Q}: \ \hat{Q}\hat{Q}^T = \hat{I} \tag{8.10}$$

Now, the deformation gradient $\hat{F}$ can be decomposed through $\hat{F} = \hat{R}\hat{U}$ by obtaining

$$U(\hat{F}) = U(\hat{Q}\hat{R}\hat{U}) \ \forall \hat{Q}: \ \hat{Q}\hat{Q}^T = \hat{I} \tag{8.11}$$

By imposing $\hat{Q} = \hat{R}^T$ we have $U(\hat{F}) = U(\hat{U})$ and, since $\hat{U}^2 = \hat{C}$, we finally obtain the dependance

$$U(\hat{F}) = U(\hat{C}) \tag{8.12}$$

where $\hat{C}$ is the right Cauchy tensor. The choice of $\hat{C}$ as an independent variable is convenient because, from its definition, $\hat{C} = \hat{F}^T\hat{F}$ is a rational function of the deformation gradient $\hat{F}$. Now we can calculate $\dot{U}$ as follows

$$\dot{U} = \frac{\partial U}{\partial C_{ij}}\dot{C}_{ij} = \frac{\partial U}{\partial C_{ij}}\left(F_{ki}\dot{F}_{kj} + \dot{F}_{ki}F_{kj}\right) \tag{8.13}$$

We remember that $\dot{F}_{kj} = L_{ks}F_{sj}$ (see Eq.(2.20)) and we obtain

$$\begin{aligned}\dot{U} &= \frac{\partial U}{\partial C_{ij}}\left(F_{ki}L_{ks}F_{sj} + L_{ks}F_{si}F_{kj}\right) \\ &= \mathrm{tr}\left[\frac{\partial U}{\partial \hat{C}}\hat{F}^T\hat{L}\hat{F} + \frac{\partial U}{\partial \hat{C}}\hat{F}^T\hat{L}^T\hat{F}\right] = \mathrm{tr}\left[2\frac{\partial U}{\partial \hat{C}}\hat{F}^T\hat{D}\hat{F}\right] \qquad (8.14)\end{aligned}$$

where $\hat{D}$ is the rate of deformation tensor defined as the symmetric part of the velocity gradient $\hat{L}$. Through the comparison of Eqs.(8.9) and (8.14) we obtain

$$\mathrm{tr}\left[\frac{\rho_0}{\rho}\hat{T}\hat{D}\right] = \mathrm{tr}\left[2\frac{\partial U}{\partial \hat{C}}\hat{F}^T\hat{D}\hat{F}\right] \qquad (8.15)$$

Further, from the commutation rule $\mathrm{tr}(\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{A})$ of the trace operation we arrive at the following relationships, which must be satisfied for any possible $\hat{D}$

$$\mathrm{tr}\left[\frac{\rho_0}{\rho}\hat{T}\hat{D}\right] = \mathrm{tr}\left[2\hat{F}\frac{\partial U}{\partial \hat{C}}\hat{F}^T\hat{D}\right] \qquad (8.16)$$

Therefore, we obtain the formal connection between the constitutive equation (giving the Cauchy stress tensor) and the strain energy function in the form

$$\hat{T} = 2\frac{\rho}{\rho_0}\hat{F}\frac{\partial U}{\partial \hat{C}}\hat{F}^T \qquad (8.17)$$

Similarly for the first Piola-Kirchhoff stress tensor we obtain

$$\hat{T}^{1\mathcal{PK}} = J\hat{T}\hat{F}^{-T} = 2\hat{F}\frac{\partial U}{\partial \hat{C}} \qquad (8.18)$$

and finally for the second Piola-Kirchhoff stress tensor

$$\hat{T}^{2\mathcal{PK}} = \hat{F}^{-1}\hat{T}^{1\mathcal{PK}} = 2\frac{\partial U}{\partial \hat{C}} \qquad (8.19)$$

We have proved that an arbitrarily nonlinear constitutive equation can be always written by means of derivations of the strain energy function: it means that the strain energy function contains the complete information about the nonlinear elastic response of a given material. For the particular case of nonlinear isotropic material the strain energy function U must depend only upon the invariants of the right Cauchy tensor $\hat{C}$. We observe that they are defined as

$$I_C = \mathrm{tr}\left[\hat{C}\right] \qquad (8.20)$$

$$II_C = \frac{1}{2}\left[\left(\mathrm{tr}\hat{C}\right)^2 - \mathrm{tr}\left(\hat{C}^2\right)\right] \qquad (8.21)$$

$$III_C = \det\hat{C} \qquad (8.22)$$

and therefore we have $U = U(I_C, II_C, III_C)$. We remember that the three invariants define the characteristic polynomial of the tensor $\hat{C}$

$$\det\left(\hat{C} - \lambda\hat{I}\right) = -\lambda^3 + \lambda^2 I_C - \lambda II_C + III_C \qquad (8.23)$$

and satisfy the so-called Cayley-Hamilton theorem

$$\hat{0} = -\hat{C}^3 + I_C\hat{C}^2 - II_C\hat{C} + III_C\hat{I} \qquad (8.24)$$

It is possible to prove that

$$\frac{\partial I_C}{\partial \hat{C}} = \hat{I}; \quad \frac{\partial II_C}{\partial \hat{C}} = I_C\hat{I} - \hat{C}; \quad \frac{\partial III_C}{\partial \hat{C}} = III_C\hat{C}^{-1}; \tag{8.25}$$

and therefore we obtain

$$\begin{aligned}\frac{\partial U(I_C, II_C, III_C)}{\partial \hat{C}} &= \frac{\partial U}{\partial I_C}\frac{\partial I_C}{\partial \hat{C}} + \frac{\partial U}{\partial II_C}\frac{\partial II_C}{\partial \hat{C}} + \frac{\partial U}{\partial III_C}\frac{\partial III_C}{\partial \hat{C}} \\ &= \frac{\partial U}{\partial I_C}\hat{I} + \frac{\partial U}{\partial II_C}\left(I_C\hat{I} - \hat{C}\right) + \frac{\partial U}{\partial III_C}III_C\hat{C}^{-1}\end{aligned} \tag{8.26}$$

This expression can be used in the Cauchy and Piola-Kirchhoff tensors given in Eqs.(8.17), (8.18) and (8.19) in order to obtain their final form in terms of the invariants of the right Cauchy tensor $\hat{C}$. Sometime the stress tensors can also be expressed in term of the Green-Lagrange strain tensor $\hat{\eta} = \frac{1}{2}\left(\hat{C} - \hat{I}\right)$; since $2d\hat{\eta} = d\hat{C}$, we have

$$\hat{T} = \frac{\rho}{\rho_0}\hat{F}\frac{\partial U}{\partial \hat{\eta}}\hat{F}^T; \quad \hat{T}^{1\mathcal{PK}} = \hat{F}\frac{\partial U}{\partial \hat{\eta}}; \quad \hat{T}^{2\mathcal{PK}} = \frac{\partial U}{\partial \hat{\eta}} \tag{8.27}$$

In this case the strain energy function U (for unit volume of the reference configuration) may be developed in power series with respect to the components of $\hat{\eta}$. This leads to the expression

$$U(\hat{\eta}) = \frac{1}{2}\mathcal{C}^{\mathcal{L}}_{ijkh}\eta_{ij}\eta_{kh} + \frac{1}{6}\mathcal{C}^{\mathcal{L}}_{ijkhnm}\eta_{ij}\eta_{kh}\eta_{nm} + ... \tag{8.28}$$

Here the $\mathcal{C}^{\mathcal{L}}_{ijkh}$ and the $\mathcal{C}^{\mathcal{L}}_{ijkhnm}$ denote the second order elastic constants (SEOC) and the third order elastic constants (TOEC), respectively (within the Lagrangian formalism).

9. The small-strain approximation

In the infinitesimal elasticity theory the extent of the deformations is assumed small. While this notion is rather intuitive, it can be formalized by imposing that for small deformations $\hat{F}$ is very similar to $\hat{I}$ or, equivalently, that $\hat{G}$ is very very similar to $\hat{I}$. It means that both $\hat{J}_L$ and $\hat{J}_E$ are very small. Therefore, we adopt as an operative definition of *small deformation* the relations

$$\mathrm{Tr}(\hat{J}_L\hat{J}_L^T) \ll 1 \ \text{ and } \ \mathrm{Tr}(\hat{J}_E\hat{J}_E^T) \ll 1 \tag{9.1}$$

i.e., a deformation will be hereafter regarded to as *small* provided that the trace of the product $\hat{J}_L\hat{J}_L^T$ or $\hat{J}_E\hat{J}_E^T$ is negligible. It means that we can assume $\hat{J}_L = \hat{J}_E = \hat{J}$ and that we can interchange the Eulerian and the Lagrangian variables without problems. Here, we write all the equations with the Eulerian variables $\vec{x}$. We observe that $\hat{J}$ can be written as the sum of a symmetric and a skew-symmetric (antisymmetric) part as follows

$$J_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{symmetric} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{skew-symmetric} = \epsilon_{ij} + \Omega_{ij} \tag{9.2}$$

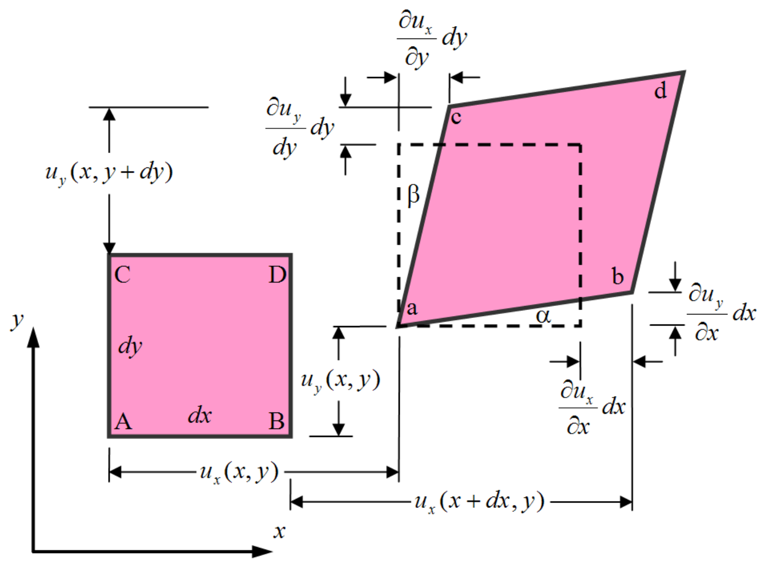

Figure 6. Two-dimensional geometric deformation of an infinitesimal material element.

The meaning of the displacement gradient can be found in Fig. 6 for a two-dimensional configuration. Accordingly, we define the (symmetric) *infinitesimal strain tensor* (or *small strain tensor*) as

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{9.3}$$

and the (antisymmetric) *local rotation tensor* as

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{9.4}$$

Such a decomposition is useful to obtain three very important properties of the small strain tensor, which is the key quantity to determine the state of deformation of an elastic body:

- for a pure local rotation (a volume element is rotated, but not changed in shape and size) we have $\hat{J} = \hat{\Omega}$ and therefore $\hat{\epsilon} = 0$. This means that the small strain tensor does not take into account any local rotation, but only the changes of shape and size (dilatations or compression) of that element of volume.
  Let us clarify this fundamental result. We consider a point $\vec{x}$ inside a volume element which is transformed to $\vec{x} + \vec{u}(\vec{x})$ in the current configuration. Under a pure local rotation we have: $\vec{x} + \vec{u}(\vec{x}) = \hat{R}\vec{x}$, where $\hat{R}$ is a given orthogonal rotation matrix (satisfying $\hat{R}\hat{R}^T = \hat{I}$). We simply obtain $\vec{u}(\vec{x}) = (\hat{R} - \hat{I})\vec{x}$ or, equivalently, $\hat{J} = \hat{R} - \hat{I}$. Since the applied deformation (i.e., the local rotation) is small by hypothesis, we observe that the difference $\hat{R} - \hat{I}$ is very small too. The product $\hat{J}\hat{J}^T$ will be therefore negligible, leading to the following expression

$$0 \cong \hat{J}\hat{J}^T = \left(\hat{R} - \hat{I}\right)\left(\hat{R}^T - \hat{I}\right) = \hat{R}\hat{R}^T - \hat{R} - \hat{R}^T + \hat{I}$$

$$= \hat{I} - \hat{R} - \hat{R}^T + \hat{I} = -\hat{J} - \hat{J}^T \tag{9.5}$$

Therefore $\hat{J} = -\hat{J}^T$ or, equivalently, $\hat{J}$ is a skew-symmetric tensor. It follows that $\hat{J} = \hat{\Omega}$ and $\hat{\epsilon} = 0$. We have verified that a pure rotation corresponds to zero strain. In addition, we remark that the local rotation of a volume element within a body cannot be correlated with any arbitrary force exerted in that region (the forces are correlated with $\hat{\epsilon}$ and not with $\hat{\Omega}$): for this reason the infinitesimal strain tensor is the only relevant object for the analysis of the deformation due to applied loads in elasticity theory.

- the infinitesimal strain tensor allows for the determination of the length variation of any vector from the reference to the current configuration. By defining ϵ_{nn} as the relative length variation in direction $\vec{n}$, we have from Table 2

$$\epsilon_{nn} = \vec{n} \cdot \hat{\epsilon}\vec{n} \tag{9.6}$$

If $\vec{n}$ is actually any unit vector of the reference frame, it is straightforward to attribute a geometrical meaning to the components ϵ_{11}, ϵ_{22}, ϵ_{33} of the strain tensor. Since $\epsilon_{nn} = \vec{e}_i \cdot (\hat{\epsilon}\ \vec{e}_i) = \epsilon_{ii}$, they describe the relative length variations along the three axes of the reference frame.

- the infinitesimal strain tensor allows for the determination of the angle variation between any two vectors from the reference to the current configuration. The variation of the angle defined by the two orthogonal directions $\vec{n}$ and $\vec{t}$ can be obtained from Table 2

$$\gamma_{nt} = 2\vec{n} \cdot \hat{\epsilon}\vec{t} \tag{9.7}$$

The present result is also useful for giving a direct geometrical interpretation of the components ϵ_{12}, ϵ_{23} and ϵ_{13} of the infinitesimal strain tensor. As an example, we take into consideration the component ϵ_{12} and we assume that $\vec{n} = \vec{e}_1$ and $\vec{t} = \vec{e}_2$. The quantity γ_{nt} represents the variation of a right angle lying on the plane (x_1, x_2). Since $\epsilon_{12} = \vec{e}_1 \cdot (\hat{\epsilon}\ \vec{e}_2)$, we easily obtain the relationship $\gamma_{nt} = 2\epsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$. In other words, ϵ_{12} is half the variation of the right angle formed by the axis x_1 and x_2. Of course, the same interpretation is valid for the other components ϵ_{23} and ϵ_{13}.

The result of the application of the small strain approximation on the main quantities of the continuum mechanics is summarized in Table 3.

Knowing the $\hat{\epsilon}$ tensor field within a strained (i.e., deformed) elastic body allows us to calculate the volume change ΔV of a given region. We get $\Delta V = \int_V \mathrm{Tr}(\hat{\epsilon})\mathrm{d}\vec{x}$, where V is the volume of the unstrained region.

The above discussion states that, given a displacement field $\vec{u}(\vec{x})$, the components of the infinitesimal strain tensor are easily calculated by direct differentiation. The inverse problem is much more complicated. Given an arbitrary infinitesimal strain tensor $\hat{\epsilon}(\vec{x})$ we could search for that displacement field $\vec{u}(\vec{x})$ generating the imposed deformation. In general, such a displacement field may not exist. There are, however, suitable conditions

Table 3. The small strain approximation.

Lagrangian vision	Eulerian vision
$\hat{J}_L = \hat{J}$	$\hat{J}_E = \hat{J}$
$\hat{F} = \hat{G}^{-1} = \hat{I} + \hat{J}$	$\hat{F}^{-1} = \hat{G} = \hat{I} - \hat{J}$
$\hat{\eta} = \hat{\epsilon}$	$\hat{e} = \hat{\epsilon}$
$\hat{C} = \hat{B} = \hat{I} + 2\hat{\epsilon}$	$\hat{C}^{-1} = \hat{B}^{-1} = \hat{I} - 2\hat{\epsilon}$
$\hat{U} = \hat{V} = \hat{I} + \hat{\epsilon}$	$\hat{U}^{-1} = \hat{V}^{-1} = \hat{I} - \hat{\epsilon}$
$\hat{R} = \hat{I} + \hat{\Omega}$	$\hat{R}^{-1} = \hat{I} - \hat{\Omega}$
$\hat{T}^{1\mathcal{PK}} = \hat{T}^{2\mathcal{PK}} = \frac{\partial U}{\partial \hat{\epsilon}}$	$\hat{T} = \frac{\partial U}{\partial \hat{\epsilon}}$

under which the solution of this inverse problem is actually found. These conditions are written in the very compact form

$$\eta_{qki}\eta_{phj}\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_h} = 0 \tag{9.8}$$

where η's are the Levi-Civita permutation symbols. Eqs.(9.8) are known as infinitesimal strain compatibility equations or Beltrami Saint-Venant equations. The balance equations assume the standard form

$$\frac{\partial T_{ji}}{\partial x_i} + b_j = \rho \frac{\partial^2 u_j}{\partial t^2} \tag{9.9}$$

$$T_{ij} = T_{ji} \tag{9.10}$$

The principles of linear and angular momentum, the definition of strain and its compatibility conditions need to be supplemented by a further set of equations, known as constitutive equations, which characterize the constitution of the elastic solid body. In the case of small deformation we can write

$$\hat{T} = \hat{T}^{1\mathcal{PK}} = \hat{T}^{2\mathcal{PK}} = \frac{\partial U}{\partial \hat{\epsilon}} \tag{9.11}$$

where the strain energy function is expressed as $U = U(\hat{\epsilon})$. Such a strain energy function U may be developed in power series with respect to the components of $\hat{\epsilon}$. This leads to the expression

$$U(\hat{\eta}) = \frac{1}{2}\mathcal{C}_{ijkh}\epsilon_{ij}\epsilon_{kh} + \frac{1}{6}\mathcal{C}_{ijkhnm}\epsilon_{ij}\epsilon_{kh}\epsilon_{nm} + ... \tag{9.12}$$

Here the $\mathcal{C}_{ijkh}$ and the $\mathcal{C}_{ijkhnm}$ denote the second order elastic constant (SEOC) and the third order elastic constant (TOEC), respectively, with reference to the small strain tensor. We can determine the relations with the elastic constants defined in Eq.(8.28): to this aim, we consider an homogeneous deformation with $\hat{F} = \hat{I} + \hat{\epsilon}$ (i.e. with $\Omega = 0$ or $\hat{J} = \hat{\epsilon}$) and we obtain $\hat{\eta} = \hat{\epsilon} + \frac{1}{2}\hat{\epsilon}^2$; so, by imposing $U(\hat{\epsilon}) = U(\hat{\eta})$ we eventually obtain

$$\mathcal{C}_{ijkh} = \mathcal{C}^{\mathcal{L}}_{ijkh} \tag{9.13}$$

$$\mathcal{C}_{ijkhnm} = \mathcal{C}^{\mathcal{L}}_{ijkhnm} + \frac{3}{2}\mathcal{C}^{\mathcal{L}}_{imkh}\delta_{jn} + \frac{3}{2}\mathcal{C}^{\mathcal{L}}_{ijkm}\delta_{hn} \tag{9.14}$$

The linear law for the relation between stress and strain is called the generalized Hooke's law. The general form of writing Hooke's law is as follows

$$T_{ij} = \mathcal{C}_{ijkh}\epsilon_{kh} \tag{9.15}$$

where $\mathcal{C}_{ijkh}$ are constants (for homogeneous materials). Eq.(9.15) is of general validity, including all the possible crystalline symmetry or, in other words, any kind of anisotropy. The tensor of the elastic constants satisfies the following symmetry rules: 1) symmetry in the first pair of indices: since $T_{ij} = T_{ji}$ we have $\mathcal{C}_{ijkh} = \mathcal{C}_{jikh}$: 2) symmetry in the last pair of indices: since $\epsilon_{kh} = \epsilon_{hk}$ we may take $\mathcal{C}_{ijkh} = \mathcal{C}_{ijhk}$; 3) symmetry between the first pair and the last pair of indices: energetic considerations leads to $\mathcal{C}_{ijkh} = \mathcal{C}_{khij}$. At the end $\mathcal{C}_{ijkh}$ has at most 21 independent components rather than the $3^4 = 81$ which, as a general fourth-rank tensor, it might have had. In the case of a linear and isotropic material we have

$$\hat{T} = \frac{E}{1+\nu}\hat{\epsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)}\hat{I}\,\mathrm{Tr}\,(\hat{\epsilon}) \tag{9.16}$$

where E and ν are the Young modulus and the Poisson ratio, respectively. We can also introduce the Lamé coefficients μ and λ as follows

$$\mu = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{9.17}$$

Therefore, Eq.(9.16) assumes the standard form

$$\hat{T} = 2\mu\hat{\epsilon} + \lambda\hat{I}\mathrm{Tr}(\hat{\epsilon}) \tag{9.18}$$

When we are dealing with a linear, isotropic and homogeneous material the governing equations of the elasticity theory can be summed up as follows

$$(\lambda+\mu)\,\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right) + \mu\vec{\nabla}^2\vec{u} + \vec{b} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{9.19}$$

This is an equation of motion where the displacement field is the single unknown, which have been called Lamé or Navier equation. Such a motion equation for a isotropic elastic body can be also written in a different form by utilizing the general property $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right) - \vec{\nabla}^2\vec{u}$, which holds for the differential operators. The result is

$$(\lambda+\mu)\,\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}\right) + (\lambda+2\mu)\vec{\nabla}^2\vec{u} + \vec{b} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{9.20}$$

Both Eq. (9.19) and Eq. (9.20) are linear partial differential equations of the second order with a vector field $\vec{u}\,(\vec{r})$ as unknown. In order to find a solution of Eq. (9.19) or Eq. (9.20) we must impose some boundary conditions depending on the physical problem under consideration. If we consider a body with an external surface $\mathtt{S}$, a first type of boundary condition fixes the values of the displacement field on this surface at any time. It means that $\vec{u} = \vec{u}(\vec{x},t)$ for any $\vec{x}\in\mathtt{S}$ and for any t in a given interval. When the entire external surface is described by these conditions we say that we are solving an elastic problem of the first kind (Dirichlet). A second kind of boundary conditions fixes the stress applied on the external surface. It means that $T_{ij}n_j = f_i(\vec{x},t)$ for any

$\vec{x} \in \mathtt{S}$ and for any t in a given interval. When the entire external surface is described by these conditions we say that we are solving an elastic problem of the second kind (Neumann). Finally, a third case can be defined by dividing the surface $\mathtt{S}$ in two parts and by applying the Dirichlet conditions to the first part and the Neumann conditions to the second part. In this case we say that we are solving an elastic problem of the third kind, subjected to mixed boundary conditions.